**Math 880** **Homework 9** **Due Wed Nov 29**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

**1.** Recall the type-A braid arrangement $\mathcal{A} = \mathcal{A}_F = \{H_{ij} \mid 1 \leqslant i < j \leqslant n\}$, consisting of the diagonal hyperplanes $H_{ij} = \{(x_1, \ldots, x_n) \in F^n \mid x_i = x_j\}$ over a field $F$. Its associated lattice $\mathcal{L}(\mathcal{A})$ comprises subspaces $U$ which are intersections of hyperplanes in $\mathcal{A}$, ordered by *reverse* inclusion, so that $\hat{0} = F^n$ and $\hat{1} = F(1, \ldots, 1)$.

The *characteristic polynomial* is: $\chi_{\mathcal{A}}(x) = \sum_{U \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, U) x^{\dim(U)}$. The complement of the hyperplanes is: $X_F = F^n - \bigcup_{H \in \mathcal{A}} H$.

**a.** Taking the arrangement $\mathcal{A}_F$ over a finite field $F = \mathbb{F}_q$, we showed $\chi_{\mathcal{A}}(q) = \#X_F$. Evaluate this by direct counting, and find the coefficients of $q^n$ in terms of familiar counting numbers on the Twelvefold Way handout.

**b.** The lattice $\mathcal{L}(\mathcal{A})$ in part (a) is isomorphic to what famililar poset $\mathcal{P}$? Determine $\mu_{\mathcal{L}(\mathcal{A})}(\hat{0}, \hat{1})$ from part (a), and compare with our known $\mu_{\mathcal{P}}$. Deduce a summation formula for the Stirling cycle number $\left[{n \atop k}\right]$. Explain this formula from elementary principles.

*Solution:*

$$\begin{aligned}
(-1)^{n-k} \left[{n \atop k}\right] &= \sum_{\dim(U)=k} \mu_{\mathcal{L}(\mathcal{A})}(\hat{0}, U) = \sum_{S \in \left\{{[n] \atop k}\right\}} \mu_{\Pi_n}(\hat{0}, S) \\
&= \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, \ldots, n_k} (-1)^{n-k} (n_1-1)! \cdots (n_k-1)! \\
\left[{n \atop k}\right] &= \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1 \cdots n_k} = \sum_{n_1+\cdots+n_k=n} \frac{\#S_n}{\#(C_{n_1} \times \cdots \times C_{n_k})}.
\end{aligned}$$

**c.** Determine the number of components of the topological space $X_{\mathbb{R}}$ using Zaslavsky's Theorem: $\pi_0(X_{\mathbb{R}}) = (-1)^n \chi_{\mathcal{A}}(-1)$. Give a representative point $(x_1, \ldots, x_n)$ lying in each connected component, with $\{x_1, \ldots, x_n\} = [n]$.

**d.** *Extra Credit:* The $n$-strand *braid group* consists of $n$-tuples of interweaving downward strands connecting $n$ fixed points at one level of $\mathbb{R}^3$ to a corresponding $n$ points at a lower level; two braids are considered identical if one can be moved to the other in $\mathbb{R}^3$ without moving the endpoints, and without passing one strand through another. Multiplication is defined as end-to-end concatenation of strands.

Consider $X_{\mathbb{C}}$, the complement of $\mathcal{A}$ in $\mathbb{C}^n$. The fundamental group $\pi_1(X_{\mathbb{C}})$ is closely related (but not isomorphic) to the braid group. Explain. How to get the braid group as the fundamental group of some space?

**2.** Recall that in a binomial poset $\mathcal{P}$, the number of maximal chains in an interval $[u, v]$ depends only on $n = \ell[u, v]$, the length of any maximal chain in $[u, v]$; and we denote the number of maximal chains by $B(n)$, the *$\mathcal{P}$-factorial function.* Set $[u, v] \sim [u', v']$ whenever $\ell[u, v] = \ell[u', v']$, and construct the reduced incidence algebra $R(\mathcal{P}) = \bigoplus_{n \geqslant 0} \mathbb{C}\, \overline{n}$, where $\overline{n} = \sum_{\ell(u,v)=n} [u, v]$ is the sum over an equivalence class. This has multiplication $\overline{n}\cdot\overline{m} = \frac{B(n+m)}{B(n)B(m)}\,\overline{n+m}$, which means $\overline{n}\mapsto\frac{x^n}{B(n)}$ gives an isomorphism $R(\mathcal{P})\cong\mathbb{C}[[x]]$. The zeta function $\zeta[u,v]=1$ corresponds to $\zeta = \sum_{n\geqslant 0}\overline{n} \cong \sum_{n\geqslant 0}\frac{x^n}{B(n)}$. Also, given a function $\alpha:\mathcal{P}\to\mathbb{C}$ for which $\alpha(u)$ depends only on the rank $n=\ell(u)=\ell[\hat{0},u]$, we naturally identify $\alpha=\sum_{n\geqslant 0}\alpha(n)\,\overline{n}\cong\sum_{n\geqslant 0}\alpha(n)\frac{x^n}{B(n)}$.

Our standard binomial poset is $\mathcal{B}_\infty$, the finite subsets $I\subset\mathbb{N}$ ordered by inclusion, having $\ell[I,J]=\#(J{-}I)$ and $B(n)=n!$. Another is $\mathcal{B}_\infty(q)$, the subspaces $U\subset F^\infty$, for $F=\mathbb{F}_q$ a finite field, ordered by inclusion. (Here $U\subset F^n$ for some $n$, and $F^\infty$ is the direct limit of $F^1\subset F^2\subset F^3\subset\cdots$.) We have $\ell[U,V]=\dim(V/U)$ and:

$$B(n) \;=\; \#\{0\lessdot U_1\lessdot\cdots\lessdot U_{n-1}\lessdot F^n\} \;=\; \#\mathrm{Flag}(F^n) \;=\; [n]_q^! \;=\; [n]_q\,[n{-}1]_q\cdots[1]_q,$$

where $[n]_q=\frac{q^n-1}{q-1}=1+q+\cdots+q^{n-1}$.

**a.** Fix $k$, and for $\#I=n$, let:

$$\alpha(I) \;=\; \alpha(n) \;=\; \#\{\text{functions}\,f:[k]\to I\} \;=\; n^k$$

$$\sigma(I) \;=\; \sigma(n) \;=\; \#\{\text{surjective}\,f:[k]\to I\} \;=\; \mathrm{surj}(k,n).$$

Every function $f:[k]\to J$ is surjective to its image $I=f([k])$, so that $\alpha(J)=\sum_{I\subset J}\sigma(I)$. Show that this imples $\alpha=\sigma\cdot\zeta$ in $R\cong\mathbb{C}[[x]]$; solve for $\sigma=\alpha\cdot\frac{1}{\zeta}$; and compute coefficients to recover the PIE formula for $\mathrm{surj}(k,n)$ from Twelvefold Way entry 3.

This is not really a new argument, just a reformulation of PIE in terms of the incidence algebra. Note that $\sigma$ is polynomial of degree $k$ in $\mathbb{C}[[x]]$, whereas $\alpha\cdot\frac{1}{\zeta}$ is a product of infinite power series.

**b.** Repeat part (a) for $\mathcal{B}_\infty(q)$, obtaining a summation formula for the number of surjective linear mappings $f:F^k\to F^n$. (Recall $\mu(n)=\mu[0,F^n]=(-1)^n q^{\binom{n}{2}}$.)

**c.** Show that the number of surjective linear mappings $f:F^k\to F^n$ is equal to the number of injective linear mappings $f:F^n\to F^k$. Determine this last number directly, obtaining a product formula much simpler than in part (b). Verify algebraically that these formulas are equal for $n=1$. *Extra Credit:* Verify the identity algebraically for $k=1$, using the $q$-Binomial Theorem.
*Solution:* For coeffs of $n\geqslant 1$, take $\prod(1-q^i x)$ for $x=-q^2$.

**3.** Multiplicative number theory analyzes the poset $\mathcal{D}_\infty$, the positive integers ordered by divisibility: $m{<}n$ means $m|n$. Define an equivalence relation on intervals: $[d,n]\sim[e,m]$ whenever $n/d=m/e$.

**a.** Is the relation $\sim$ stricter, looser, or the same as poset isomorphism $\cong$?

**b.** Show $\mathcal{D}_\infty$ is *not* a binomial poset by finding intervals with $\ell[1,n]=\ell[1,m]$, but with different numbers of maximal chains.

**c.** Nevertheless, we can construct the reduced incidence algebra $R=R(\mathcal{D}_\infty)=\{\sum_{n\geqslant 1}a_n\overline{n}\text{ for }a_n\in\mathbb{C}\}$, where $\overline{n}=\sum_{i\geqslant 1}[i,in]$ is the sum over an equivalence class of intervals.

Show that $\overline{n}\mapsto n^{-s}$, for $s$ a formal variable, defines an isomorphism from $R$ to the ring of formal Dirichlet series $\alpha(s)=\sum_{n\geqslant 1}\frac{a_n}{n^s}$, with the natural series multiplication. Knowing the invertible elements in the incidence algebra $R$, characterize the invertible elements in the Dirichlet series ring.

**d.** The zeta function $\zeta\in R$ corresponds to the classical Riemann zeta function

$\zeta(s) = \sum_n \frac{1}{n^s}$. For which $s \in \mathbb{C}$ is $\zeta(s)$ an absolutely convergent series? Write the reciprocal $1/\zeta(s)$ as an explicit Dirichlet series; does it converge for the same $s$? (Riemann determined the analytic continuation of $\zeta(s)$, and the Riemann Hypothesis concerns the location of the zeroes $\zeta(s) = 0$.)

**e.** A multiplicative arithmetic function $\alpha : \mathcal{D}_\infty \to \mathbb{Z}$ satisfies $\alpha(nm) = \alpha(n)\,\alpha(m)$ whenever $\gcd(n,m) = 1$. The unique factorization of natural numbers into prime powers, $n = \prod_i p_i^{\ell_i} = 2^{\ell_1} 3^{\ell_2} 5^{\ell_3} \cdots$, gives isomorphisms $\mathcal{D}_\infty \cong \mathbb{N} \times \mathbb{N} \times \cdots$ and $R(\mathcal{D}_\infty) \cong R(\mathbb{N}) \times R(\mathbb{N}) \times \cdots$. Use this to give a product formula (Euler product) for $\sum_n \frac{\alpha(n)}{n^s}$; in particular for $\zeta(s)$ and $1/\zeta(s)$. Hint: For $R(\mathbb{N}) \cong \mathbb{C}[[x]]$, substitute $x = x_i = p_i^{-s}$.

**f.** Express the Dirichlet series $\sum_n \frac{\phi(n)}{n^s}$ in terms of $\zeta(s)$, where $\phi(n)$ is the Euler phi-function (totient function) from HW 6 #4 and HW 8 #4. Also, give an explicit formula for the coefficients $d(n)$ of the Dirichlet series $\zeta(s)^2 = \sum_n \frac{d(n)}{n^s}$. Give Euler products for both of these series.