Math 880 **Homework 8** **Due Mon Nov 13**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

1. For a poset $\mathcal{P}$, we define the Möbius function $\mu = \mu_{\mathcal{P}}$ by the equations: $\mu[a,a] = 1$ and $\sum_{a \leqslant c \leqslant b} \mu[a,c] = 0$ for $a < b$. For a direct product poset $\mathcal{P} \times \mathcal{Q}$, with $(p,q) \leqslant (p',q')$ whenever $p \leqslant p'$ and $q \leqslant q'$, prove that the Möbius function is the product: $\mu_{\mathcal{P}\times\mathcal{Q}}[(p,q),(p',q')] = \mu_{\mathcal{P}}[p,p']\,\mu_{\mathcal{Q}}[q,q']$.

2. For the poset $\mathcal{P} = \mathcal{D}_{12}$, the 6-element poset of divisors of 12 ordered by divisibility, work out the Möbius function $\mu[a,b]$ in several ways:

a. Write a 6×6 matrix Z corresponding $\zeta[a,b] = 1$ for all $a \leqslant b$, and invert by Gaussian elimination: write a double matrix $[Z \,|\, I\,]$, then row reduce to the form $[\,I \,|\, M]$, so that $M = Z^{-1}$.

b. Write $Z = I + N$, where I is the identity matrix and N is strictly upper-triangular (nilpotent with $N^6 = 0$), and compute (by hand or computer) a geometric series for $M = (I+N)^{-1}$.

c. Use the recurrence $\mu(a,a) = 1$, $\mu(a,b) = -\sum_{a \leqslant c < b} \mu(a,c)$ for each $a \in \mathcal{P}$, computed by drawing it on a different copy of the Hasse diagram for each a. *Hint:* Draw the Hasse diagram of $\mathcal{D}_{12} \cong [2]\times[3]$ as a rectangle.

d. Apply the product formula of #1 above to $\mathcal{D}_{12} \cong [2]\times[3]$. Match this with Möbius' original description: $\mu(d,n) = (-1)^k$ if n/d is the product of k distinct primes, and $\mu(d,n) = 0$ if n/d is divisible by a square number.

e. Evaluate Phillip Hall's Formula: $\mu(\hat{0},\hat{1}) = \hat{c}_0 - \hat{c}_1 + \hat{c}_2 - \cdots$, where $\hat{0} = 1$, $\hat{1} = 12$, and $\hat{c}_d$ is the number of chains of length d from $\hat{0}$ to $\hat{1}$ in $\mathcal{P}$.

f. Consider $\mathcal{P} = \mathcal{Q} \sqcup \{\hat{0},\hat{1}\}$, where $\mathcal{Q}$ is the open interval $(\hat{0},\hat{1})$, and form the simplicial complex $\Delta(Q) \subset \wp[4]$ whose elements are chains in $\mathcal{Q}$. Draw a picture of the corresponding topological space, consisting of intervals (one-simplexes) glued at endpoints.

Hall's Formula says $\mu[\hat{0},\hat{1}] = \tilde{\chi}(\Delta(Q))$, the reduced Euler characteristic of the above topological space, the alternating sum of the number of simplexes of each dimension, minus 1. Compute $\tilde{\chi}(\Delta(Q))$ from this definition. Also, find the simplest triangulation of this space, and compute $\tilde{\chi}$ from that.

g. Generalizing (d), for any interval $\mathcal{P}' = [a,b] \subset \mathcal{P}$, we have the corresponding open interval $\mathcal{Q}' = (a,b)$. For each $[a,b]$, draw the space of the simplicial complex $\Delta(\mathcal{Q}')$, and count chains (or simplexes) to find $\tilde{\chi}(\Delta(\mathcal{Q}')) = \mu[a,b]$.

Extra Credit: The space $\Delta(Q')$ has a topological meaning in terms of the full space $\Delta(\mathcal{P})$. Stanley Ch 3.8, Eq (3.26), p 271 defines the *link* of a simplex $F \in \Delta$, as the set of simplexes G which are disjoint from F, but linked to it by a larger simplex:

$$\mathrm{link}_{\Delta}(F) = \{G \in \Delta \mid G \cap F = \varnothing,\, G \cup F \in \Delta\}.$$

Given $\mathcal{Q}' = (a,b)$, let $F = \{\hat{0} \lessdot a_1 \lessdot \cdots \lessdot a < b \lessdot b_1 \lessdot \cdots \lessdot \hat{1}\}$. Then we have $\Delta(\mathcal{Q}) = \mathrm{link}_{\Delta(P)}(F)$.

PROBLEM: Picture an example of this. Perhaps make a paper or computer model of $\Delta(\mathcal{P})$, which consists of 3 tetrahedra (3-simplexes) glued together.

3. For each of the following lattices $\mathcal{L}$, give an explicit description of covering relations $a \lessdot b$, meet $a \vee b$, and join $a \wedge b$. If $\mathcal{L}$ is not a distributive lattice, give an example of the failure of each distributive law. If $\mathcal{L}$ is distributive, describe its sub-poset $\mathcal{P}$ of join-irreducibles.

a. D_n, the whole-number divisors of n under divisibility

b. $B_n(q)$, the subspaces of $\mathbb{F}_q^n$ under inclusion

c. Π_n, unordered set partitions of $[n]$ under refinement: $\hat{0} = \{1 \,|\, 2 \,|\, \cdots \,|\, n\}$.

3c. Solution: A partition S in Π_n corresponds to an equivalence relation $\overset{S}{\sim}$ on $[n]$. Then $S \wedge T$ corresponds to the equivalence relation $i \overset{S\wedge T}{\sim} j$ whenever $i \overset{S}{\sim} j$ and $i \overset{T}{\sim} j$. Also, $S \vee T$ corresponds to the equivalence relation generated by $\overset{S}{\sim}$ and $\overset{T}{\sim}$ together. That is, $(i \overset{S}{\sim} j$ or $i \overset{T}{\sim} j) \Longrightarrow i \overset{S\vee T}{\sim} j$, and take the transitive closure.

4. The Euler totient function is $\phi(n) = \#\{j \leqslant n \mid \gcd(j,n) = 1\}$. Claim:

$$\phi(n) = \sum_{d|n} \mu(n/d)\, d.$$

Here $\mu(n/d) = \mu(d,n)$, the Möbius function of $\mathcal{D}_n$.

Check this formula for $n = 12$. Prove it in general, using Möbius inversion and the elementary theory of divisibility.

Solution: Main point: $\sum_{d|n} \phi(d) = n$.

Pf 1: $\#\{a \leqslant n \mid \gcd(a,n) = d\} \xrightarrow{\div d} \{b \leqslant \frac{n}{d} \mid \gcd(b, \frac{n}{d}) = 1\}$

Pf 2: For $n = p_1^{m_1} \cdots p_r^{m_r}$, note $\phi(ab) = \phi(a)\phi(b)$ for $\gcd(a,b) = 1$, and $\phi(p^k) = p^k - p^{k-1}$. Thus:

$$\begin{aligned}
\sum_{d|n} \phi(d) &= \sum_{k_1=0}^{m_1} \cdots \sum_{k_r=0}^{m_r} \phi(p_1^{k_1}) \cdots \phi(p_r^{k_r}) \\
&= \left(\sum_{k_1=0}^{m_1} \phi(p_1^{k_1}) \right) \cdots \left(\sum_{k_r=0}^{m_r} \phi(p_1^{k_r}) \right) \\
&= \left(\sum_{k_1=0}^{m_1} (p_1^{k_1} - p_1^{k_1-1}) \right) \cdots \left(\sum_{k_r=0}^{m_r} (p_r^{k_r} - p_i^{k_r-1}) \right) \\
&= p_1^{m_1} \cdots p_r^{m_r} \;=\; n.
\end{aligned}$$