Math 880 **Homework 7** **Due Mon 10/30**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

NOTES: The Grassmannian $\mathrm{Gr}(k, F^n)$ is the parameter space whose points correspond to k-dimensional subspaces V in the n-dimensional vector space F^n over a given field F. We specify $V = \mathrm{Span}_F(v_1, \ldots, v_k)$ by a $k\times n$ matrix of row vectors, with change-of-basis symmetry group $\mathrm{GL}_k(F)$. This matrix can be normalized by making a given $k \times k$ submatrix into the identity, in columns $I = \{i_1 < \cdots < i_k\}$, provided the determinant of this submatrix is nonzero:

$$V \;=\; \mathrm{GL}_k \circlearrowright \begin{bmatrix} \text{------}\; v_1 \;\text{------} \\ \text{------}\; v_2 \;\text{------} \\ \vdots \\ \text{------}\; v_k \;\text{------} \end{bmatrix} \;=\; \begin{bmatrix} *\ *\cdots 1\cdots 0\cdots 0\cdots *\ * \\ *\ *\cdots 0\cdots 1\cdots 0\cdots *\ * \\ \vdots \\ *\ *\cdots 0\cdots 0\cdots 1\cdots *\ * \end{bmatrix}$$

The $*$'s denote $k(n-k)$ free parameters in F defining a coordinate chart U_I of the Grassmannian, making it into an F-manifold: $\mathrm{Gr}(k, F^n) = \bigcup_I U_I$. We define the Schubert cell decomposition $\mathrm{Gr}(k, F^n) = \coprod_I X_I$ by letting X_I consist of those $V \in U_I$ which have no $*$'s to the right of any 1 (row-echelon form). A geometric defining condition for $V \in X_I$ is: $\dim(V \cap E_r) = \#(I \cap [r])$ for $r = 1, \ldots, n$, where $E_r = \mathrm{Span}(e_1, \ldots, e_r)$ and $e_1, \ldots, e_n$ is the standard basis. The topological closure $\overline{X}_I$ is given by: $\dim(V \cap E_r) \geqslant \#(I \cap [r])$ for $r = 1, \ldots, n$. Containment of cell closures defines the Bruhat degeneration order on index sets: $I \overset{\deg}{\leqslant} J$ whenever $X_I \subset \overline{X}_J$.

EXAMPLE: For $\mathrm{Gr}(2, F^4)$, we have:

$$U_{34} \;=\; X_{34} \;=\; \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix} \;=\; \{V \mid \dim(V \cap E_3) = 1\,, V \cap E_2 = 0\},$$

$$U_{14} \;=\; \begin{bmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{bmatrix}, \qquad X_{14} \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix} \;=\; \{V \mid E_1 \subset V \not\subset E_3\}.$$

Here are the cell closures with defining conditions, height indicating degeneration order:

$$\overline{X}_{34} = \mathrm{Gr}(2, F^4)$$
$$\overline{X}_{24} = (\dim(V \cap E_2) \geqslant 1)$$
$$\overline{X}_{23} = (V \subset E_3) \qquad\qquad \overline{X}_{14} = (E_1 \subset V)$$
$$\overline{X}_{13} = (E_1 \subset V \subset E_3)$$
$$\overline{X}_{12} = (V = E_2)$$

To verify $\{1,3\} \overset{\deg}{\leqslant} \{1,4\}$, we show that any plane $V_\circ \in X_{13}$ is approached by planes in X_{14}: we find a continuous family $\mathcal{V} : F \to \mathrm{Gr}(2, F^4)$ with $\mathcal{V}(t) \in X_{14}$ for $t \neq 0$, and $\mathcal{V}(0) = V_\circ$:

$$V_\circ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 \end{bmatrix}, \qquad \mathcal{V}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a/t & 1/t & 1 \end{bmatrix} \text{ for } t \neq 0.$$

Similarly, the flag manifold $\mathrm{Fl}(F^n)$ is the parameter space of flags $V_\bullet = (V_1 \subset \cdots \subset V_{n-1})$, where V_k is a k-dimensional subspace of F^n. We specify $V_\bullet$ by an an ordered basis $(v_1, \ldots, v_n)$ of F^n, with $V_k = \mathrm{Span}_F(v_1, \ldots, v_k)$; the basis forms an $n\times n$ matrix of row vectors. The change-of-basis symmetry group B of $V_\bullet$ consists of all lower-triangular matrices in $\mathrm{GL}_n(F)$, since we can add a multiple of v_i only to a later basis vector to leave each V_k invariant. We get a Schubert cell decomposition indexed by permutations $w \in S_n$: $\mathrm{Fl}(F^n) = \bigcup_w X_w$, where X_w consists of $V_\bullet$ whose B-reduced form is a permutation matrix w, plus $*$ coordinates in the positions of the Röthe diagram $D(w) = \{(i,j) \mid j < w(i),\ i < w^{-1}(j)\}$. Thus $\dim(X_w) = \#D(w)$.

1a. Determine the Gaussian binomial coefficient $\left[{6 \atop 3}\right]_q = \#\mathrm{Gr}(3,\mathbb{F}_q^6)$ as the number of $3{\times}6$ V-basis matrices divided by the number of $3{\times}3$ change-of-basis matrices; also as a quotient of q-integers $[n]_q = \frac{q^n-1}{q-1}$; and finally as a polynomial by dividing through (with computer).

b. There are $\binom{6}{3} = 20$ sets $I = \{i_1, i_2, i_3\} \subset [6]$ indexing the Schubert cells, in bijection with partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0)$ with $\lambda_1 \leqslant 6{-}3$: the Young diagrams fit in $3{\times}(6{-}3)$. List all I's and λ's, along with the size measure $q^{|I|} = q^{|\lambda|} = \#X_I$. Compare with part (a).

2a. Verify the q-Binomial Theorem:

$$\prod_{i=1}^{n}(1{+}q^i x) \;=\; \sum_{k=0}^{n} q^{\binom{k+1}{2}} \left[{n \atop k}\right]_q x^k$$

for the special case $n = 3$. Multiply out by hand!

b. Consider the bigraded class of all $I \subset [3]$ with size measures $\Sigma(I) = \sum_{i\in I} i$ and $\#I$. Write the bivariate generating function $F(x) = \sum_{I\subset[3]} q^{\Sigma(I)} x^{\#I}$ directly as a sum over all I, and also using the bivariate Product Principle, transforming sets into bit-strings. Explain why this is equivalent to our Grassmannian counting formulas $\left[{3 \atop k}\right]_q = \sum_I q^{|I|}$ over $I \subset [3]$, $\#I = k$, where $|I| = \Sigma(I) - \binom{k+1}{2}$.
REVISE: *Prove* the q-Binomial Theorem using the Prob 1 expansion of $\left[{n \atop k}\right]_q$.
ALSO: Negative exponent q-Binomial Theorem:

$$\sum_{k\geqslant 0} \frac{[n]_q^{\overline{k}}}{[k]_q^!} x^k = \prod_{i=1}^{n} \frac{1}{1 - q^{i-1}x}.$$

c. Generalize the q-Binomial Theorem to an infinite product:

$$\prod_{i=1}^{\infty}(1{+}q^i x) \;=\; \sum_{I} a_k(q) x^k,$$

where the right summation runs over finite subsets $I \subset \mathbb{Z}_{>0}$, and the coefficients are certain rational functions $a_k(q) \in \mathbb{C}(q) \subset \mathbb{C}[[q]]$. *Hint*: To find $a_k(q)$, transform $I = \{i_1 {<} \ldots {<} i_k\}$ into a partition $\lambda = (\lambda_1 {\geqslant} \cdots {\geqslant} \lambda_k {\geqslant} 0)$, then further into exponential notation $1^{m_1} 2^{m_2} \cdots$, recording multiplicities $m_j = \#\{i \mid \lambda_i = j\}$.

d. Repeat part (c) for the product: $\prod_{i=1}^{\infty}(1{-}q^i x)^{-1}$. *Hint:* The transformation is simpler in this case, since a multiset $I = \{i_1 {\leqslant} \cdots {\leqslant} i_k\}$ is already the same thing as a partition.

3a. For each Schubert cell $X_w \subset \mathrm{Fl}(F^3)$ corresponding to $w \in S_3$, write the B-reduced matrix form corresponding to the Röthe diagram $D(w)$. For $F = \mathbb{F}_q$, explicitly verify:

$$\#\mathrm{Fl}(\mathbb{F}_q^3) \;=\; \frac{\#\mathrm{GL}_3(\mathbb{F}_q)}{\#B} \;=\; [3]_q[2]_q[1]_q \;=\; \sum_{w\in S_3} q^{\mathrm{inv}(w)},$$

with the inversion measure $\mathrm{inv}(w) = \#\{(i,j) \mid i{<}j\,,\ w(i) > w(j)\} = \#D(w) = \dim X_w$.

b. For each Schubert cell closure $\overline{X}_w$, describe its flags $V_\bullet = (V_1 {\subset} V_2)$ in terms of their relation to the standard flag $E_1 \subset E_2$. For example, the one-point minimal cell closure is: $\overline{X}_{123} = \{E_\bullet\} = \{V_\bullet \mid V_1 = E_1\,,\ V_2 = E_2\}$. Arrange these according their Bruhat degeneration order: $w \overset{\deg}{\leqslant} u$ whenever $\overline{X}_w \subset \overline{X}_u$.

c. The moves $w' \overset{\mathrm{mv}}{<} w$ defining the combinatorial Bruhat order correspond to minimal containments $\overline{X}_{w'} \subset \overline{X}_w$. Define each relation in terms of moving a certain pair of 1's in the permutation matrix of w to get w', and conjecture a general move rule. For each minimal containment and any $(V_\bullet) \in X_{w'}$, give a family $\mathcal{V} : F \to \mathrm{Fl}(F^n)$ with $\mathcal{V}(t) \in X_w$ for $t \neq 0$ and $\mathcal{V}(0) = V_\bullet$.

Alternative Challenge Problems. You may substitute one of the following for one of #1,2,3 above.

A-1. Define a combinatorial Bruhat order $w \overset{\text{co}}{\leqslant} y$, a non-recursive criterion purely in terms of the permutations $w, y \in S_n$, which coincides with the degeneration order $w \overset{\text{deg}}{\leqslant} y$. Prove as much as you can about it, similarly to the Bruhat order for Grassmannians.

A-2. Let $V, W \subset \mathcal{F}_q^n$ be subspaces of dimension $\dim(V) = k$ and $\dim(W) = \ell$ which are transverse, meaning $V \cap W = \{0\}$. For fixed k, ℓ, n, find a formula for the number of such pairs (V, W). Hint: The answer is a power of q times a q-multinomial coefficient.