Math 880 **Homework 6** **Due Fri 10/20**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

NOTES: For a group G acting on a set $\mathcal{A}$, Burnside's Lemma counts the orbits $\bar{\mathcal{A}} = \mathcal{A}/G$ as:

$$\#\bar{\mathcal{A}} \;=\; \frac{1}{\#G}\sum_{g\in G}\#\mathcal{A}_g\,,$$

the average size of the fixed point sets $\mathcal{A}_g = \{a \in \mathcal{A} \mid g.a = a\}$. This is proved by two ways of counting the stabilizer set $\mathcal{S} = \{(g,a) \in G\times\mathcal{A} \mid g.a = a\}$ as:

$$\#\bar{\mathcal{A}} \;=\; \sum_{a\in\mathcal{A}}\frac{1}{(\#G.a)} \;=\; \sum_{a\in\mathcal{A}}\frac{\#G_a}{\#G} \;=\; \frac{\#\mathcal{S}}{\#G} \;=\; \frac{1}{\#G}\sum_{g\in G}\mathcal{A}_g,$$

since the size of each orbit $G.a$ is $(\#G)/(\#G_a)$.

For a group of permutations $G \subset S_k$, we define the *cycle index polynomial*:

$$Z_G(u_1,\ldots,u_k) \;=\; \sum_{w\in G} u_1^{c_1(w)}\cdots u_k^{c_k(w)},$$

where $c_i(w) = \#$ i-cycles of w. For example, $G = S_3$ has $Z_G = u_1^3 + 3u_1u_2 + 2u_3$.

For a class of functions $\mathcal{F} \subset \{f : [k] \to [n]\}$ the pattern inventory polynomial is the multivariate generating function:

$$P_{\mathcal{F}}(x_1,\ldots,x_n) \;=\; \sum_{f\in\mathcal{F}} x_f \quad \text{where} \quad x_f = x_{f(1)}\cdots x_{f(k)} = x_1^{\#f^{-1}(1)}\cdots x_n^{\#f^{-1}(n)}.$$

For a group $G \subset S_k$ which acts on $\mathcal{F}$ by permuting the domain $[k]$, Polya's Theorem gives the pattern inventory of the quotient $\bar{\mathcal{F}} = \mathcal{F}/G$:

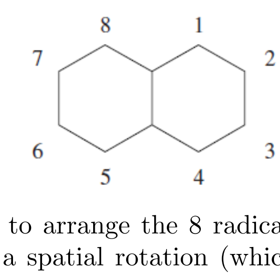
$$P_{\overline{\mathcal{F}}}(x_1,\ldots,x_n) \;=\; \tfrac{1}{\#G}\sum_{w\in G} P_{\mathcal{F}_w}(x_1,\ldots,x_n) \;=\; \tfrac{1}{\#G}Z_G(p_1,\ldots,p_k).$$

Here the first expression uses the fixed-point set $\mathcal{F}_w$, the functions which are constant on each cycle of w in $[k]$:

$$\mathcal{F}_w \;=\; \{f\in\mathcal{F} \mid w.f = f\} \;=\; \{f : \{\text{cycles of } w\} \to [n]\},$$

and the second expression uses the power symmetric polynomial $p_i(x_1,\ldots,x_n) = x_1^i + \cdots + x_n^i$.

1. Tetramethylnaphthalene is an organic molecule with carbon skeleton shown below, with 4 of the positions 1–8 filled with methyl radicals CH_3, and the other 4 filled with hydrogen radicals H.



There are several isomers (ways to arrange the 8 radicals), but two isomers are physically the same if they are related by a spatial rotation (which might look like a reflection on a plane diagram). Determine the number of distinct isomers: find the natural symmetry group $G \subset S_8$, compute its cycle index polynomial, and expand the pattern inventory polynomial via computer algebra.

2. An action of a group G on an unlabeled graded class $\mathcal{F}$ preserves size: $|w.f| = |f|$. By Burnside's Theorem, the (ordinary) generating function of the G-orbits $\bar{\mathcal{F}} = \mathcal{F}/G$ is the average of the generating functions of the fixed classes: $\overline{F}(x) = \frac{1}{\#G}\sum_{w\in G} F_w(x)$. In particular, let $\mathcal{F} = \{f : [k] \to \mathcal{A}\} \cong \mathcal{A}^k$ be functions from $[k]$ to a graded class $\mathcal{A}$, with $|f| = \sum_{i=1}^k |f(i)|$, and let $G \subset S_k$ permute $[k]$. Again, the fixed-point set $\mathcal{F}_w$ consists of $f : \{\text{cycles of } w\} \to \mathcal{A}$.

a. Use the graded counting principles to show that the generating function of $\mathcal{F}_w$ is: $F_w(x) = A(x)^{c_1(w)} A(x^2)^{c_2(w)} \cdots A(x^k)^{c_k(w)}$, and $\overline{F}(x) = \frac{1}{\#G} Z(A(x), A(x^2), \ldots, A(x^k))$.

b. Compute the generating function of $\mathrm{MS}\textsc{et}_4\,\mathcal{A} \cong \mathcal{A}^4/S_4$, the class of multisets comprising 4 elements in $\mathcal{A}$.

3. Define $\mathcal{F} = \mathrm{MS}\textsc{et}\,\mathcal{A} = \coprod_{k\geq 0} \mathrm{MS}\textsc{et}_k\,\mathcal{A} \cong \{m : \mathcal{A} \to \mathbb{N}\}$, the class of multisets of $k \geq 0$ elements in $\mathcal{A} = \coprod_{i\geq 0} \mathcal{A}_i$, which can be realized as multiplicity functions m with $m(a) > 0$ for only finitely many $a \in \mathcal{A}$. This is an unlabeled bigraded class with size $|m| = \sum_{a\in\mathcal{A}} m(a)\,|a|$ and weight $\mathrm{wt}(m) = \sum_{a\in\mathcal{A}} m(a) = k$. The bivariate generating function can be computed by the graded Product Principle:

$$F(x,t) \;=\; \sum_{m\in\mathcal{F}} x^{|m|} t^{\mathrm{wt}(m)} \;=\; \prod_{i\geq 1} (1 - t x^i)^{-A_i}.$$

a. Manipulate the equation $F = \exp(\log F)$ to show that:

$$F(x,t) \;=\; \exp\!\left(A(x)t + A(x^2)\frac{t^2}{2} + A(x^3)\frac{t^3}{3} + \cdots \right).$$

b. Conclude that the generating function of $\mathrm{MS}\textsc{et}_k\,\mathcal{A}$ is the t^k coefficient of the above power series. Find this for $k = 4$ by computer, and compare with #2b above.

c. Repeat (a) and (b) above for $\mathcal{F} = \mathrm{Set}\,\mathcal{A}$, the bigraded class of all finite subsets of $\mathcal{A}$. Is there also a Polya Theory way to get this formula?

4. Define the *Euler phi-function* $\phi(k) = \#\{j \leq k \mid \gcd(j,k) = 1\}$, the number of integers in $[1,k]$ which are relatively prime to k. For example, $\phi(q) = q{-}1$ for q prime.

a. Use Inclusion-Exclusion to prove that $\phi(k) = \Pi_i\, q_i^{\ell_i - 1}(q_i{-}1)$, where we have the prime factorization $k = q_1^{\ell_1} q_2^{\ell_2} \cdots$.

b. Determine the cycle index $Z_G(u_1, \ldots, u_k)$ for $G = C_k$, the cyclic group generated by a k-cycle such as $(12\cdots k) \in S_k$. The answer will involve the phi-function.

c. Consider the class $\mathcal{F} = \mathrm{C}\textsc{yc}_k\,\mathcal{A}$ consisting of all cyclic arrangements of k elements of $\mathcal{A}$. Show that the generating function is: $F(x) \;=\; \frac{1}{k}\sum_{d|k} \phi(d) A(x^d)^{k/d}$.

5. Let $\mathrm{irr}_n(p)$ be the number of irreducible polynomials $f(x) \in \mathbb{F}_p[x]$ whose degree d is a divisor of n, i.e. $d|n$. Galois theory tells us this is equal to the number of orbits of the cyclic Galois group $G = \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) = \{1, \Phi, \ldots, \Phi^{n-1}\}$ acting on $\mathbb{F}_q$, where $q = p^n$ and $\Phi(\alpha) = \alpha^p$ is the Frobenius automorphism. (In fact, each orbit is the set of roots of an irreducible polynomial.) Use Burnside's Lemma to find a formula for $\mathrm{irr}_n(p)$, using the phi-function.

Hint: Apply Burnside's Lemma $\#\mathbb{F}_q/G = \frac{1}{\#G}\sum_{g\in G} \#(\mathbb{F}_q)_g$. The *degree* of a field extension $K \supset L$ is the dimension of K as a vector space over the field L. The Galois group $G = \mathrm{Gal}(K/L) = \{\sigma : K \xrightarrow{\sim} K \mid \sigma(a) = a \text{ for } a \in L\}$ contains the field automorphisms of K which fix L. The Galois Correspondence states that a d-element subgroup $H \subset G$ has fixed field $K_H \supset L$ of degree n/d.