Math 880 **Homework 5** **Due Mon 10/9**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

1. Recall the constructons of labeled graded classes $\tilde{\mathcal{C}}$ which allow us to compute their exponential generating functions $\tilde{C}(x) = \sum_{k\geq 0} \tilde{C}_k \frac{x^k}{k!}$:

$$\begin{array}{ll} \tilde{\mathcal{C}} = \tilde{\mathcal{A}} * \tilde{\mathcal{B}}\ ,\ \tilde{C}(x) = \tilde{A}(x)\tilde{B}(x) & \tilde{\mathcal{C}} = \tilde{\mathcal{A}}^{\min} * \tilde{\mathcal{B}}\ ,\ \tilde{C}(x) = \int \tilde{A}'(x)\tilde{B}(x)\,dx \\ \tilde{\mathcal{C}} = \textsc{Seq}_n\tilde{\mathcal{A}}\ ,\ \tilde{C}(x) = \tilde{A}(x)^n & \tilde{\mathcal{C}} = \textsc{Seq}\,\tilde{\mathcal{A}}\ ,\ \tilde{C}(x) = 1/(1-\tilde{A}(x)) \\ \tilde{\mathcal{C}} = \textsc{Set}_n\tilde{\mathcal{A}}\ ,\ \tilde{C}(x) = \tilde{A}(x)^n/n! & \tilde{\mathcal{C}} = \textsc{Set}\,\tilde{\mathcal{A}}\ ,\ \tilde{C}(x) = \exp\tilde{A}(x) \\ \tilde{\mathcal{C}} = \textsc{Cyc}_n\tilde{\mathcal{A}}\ ,\ \tilde{C}(x) = \tilde{A}(x)^n/n & \tilde{\mathcal{C}} = \textsc{Cyc}\,\tilde{\mathcal{A}}\ ,\ \tilde{C}(x) = -\log(1-\tilde{A}(x)) \end{array}$$

(We cannot construct labeled multisets: distinct labels prevent exact copies within an object.)

a. Consider the class $\tilde{\mathcal{D}}$ of derangements from HW 3 #3a: that is, $\tilde{\mathcal{D}}_k$ is the class of permutations $w \in S_k$ with $w(i) \neq i$ for all i. Construct $\tilde{\mathcal{D}}$, find its generating function $\tilde{D}(x)$, and extract the formula for D_k which we previously obtained by inclusion-exclusion.

b. Show that the number e_{2k} of permutations in S_{2k} which have only even-length cycles is $(1\cdot 3\cdot 5\cdots(2k-1))^2$. For example, $e_4 = (1\cdot 3)^2 = 9$ counts:

$$(1234), (1243), (1324), (1342), (1423), (1432), (12)(34), (13)(24), (14)(23).$$

Hint: The even part of a function $f(x)$ is $\frac{1}{2}(f(x) + f(-x))$.
Extra Credit: Give a bijective transformation proof of this formula.

2. Recall the constructions of unlabeled classes $\mathcal{C}$ which allow us to compute their ordinary generating functions $C(x) = \sum_{k\geq 0} C_k x^k$, starting with $\mathcal{C} = \mathcal{A}\times\mathcal{B} \ \Rightarrow\ C(x) = A(x)B(x)$:

$$\begin{array}{ll} \mathcal{C} = \textsc{Seq}_n\mathcal{A}\ ,\ C(x) = A(x)^n & \mathcal{C} = \textsc{Seq}\,\mathcal{A}\ ,\ C(x) = 1/(1-A(x)) \\ \mathcal{C} = \textsc{Set}\,\mathcal{A}\ ,\ C(x) = \prod_{i\geq 1}(1+x^i)^{A_i} & \mathcal{C} = \textsc{MSet}\,\mathcal{A}\ ,\ C(x) = \prod_{i\geq 1}(1-x^i)^{-A_i} \end{array}$$

Problem: Let $\mathcal{P}_k^{\mathrm{odd}}$ be the partitions of k into parts which are odd numbers: $p^{\mathrm{odd}}(6) = 4$ counts $5+1 = 3+3 = 3+1+1 = 1+1+1+1+1$. Let $\mathcal{Q}_k$ be the partitions of k into distinct parts: $q(6) = 4$ counts $6 = 5+1 = 4+2 = 3+2+1$. Find the ordinary generating functions $P^{\mathrm{odd}}(x)$ and $Q(x)$, and show that they are equal, which implies $p^{\mathrm{odd}}(k) = q(k)$ for all k. *Hint:* A partition is a multiset of numbers. What about convergence of infinite products?

3. Use generating functions to explicitly count the types of rooted trees listed below.

Example: Recall the class $\mathcal{B}_n$ of plane binary trees on n unlabeled vertices: starting with the root, each vertex has no children, or a left and a right child. We set $\mathcal{B}_0 = \{\}$. A Deletion Transform gives the recursive construction $\mathcal{B} \cong [1]\times\mathcal{B}\times\mathcal{B} \sqcup [1]$, and the equation $B(x) = xB(x)^2 + x$, or $B(x)/(1+B(x)^2) = x$, so that $B(x)$ is the inverse function of $x/\Phi(x)$ for $\Phi(x) = 1+x^2$. Lagrange Inversion then gives, for $n = 2k+1$:

$$B_n \ = \ \frac{1}{n}[x^{n-1}]\Phi(x)^n \ = \ \frac{1}{n}\binom{n}{k} \ = \ \frac{1}{k+1}\binom{2k}{k} \ = \ C_k, \text{ a Catalan number.}$$

a. Plane trees: n unlabeled vertices, any number of children in an ordered sequence.

b. Labeled plane trees: n labeled vertices, any number of children in an ordered sequence.

c. Ternary trees: n unlabeled vertices, zero or three children in an ordered sequence.

d. Increasing trees: n labeled vertices, any number of children in an unordered set. Also, each child must have a label larger than its parent (so the root must be 1).
Extra Credit: Bijectively prove the simple formula for (d).

4. Recall that, given a generating function $A(x) = \sum_{k\geq 0} A_k x^k$ which has been evaluated as an explicit analytic formula, $A(x) = f(x)$ with $f(0) \neq 0$, we can obtain a recurrence for the coefficients A_k using the Dlog Method. We take the logarithmic derivative:

$$x\frac{d}{dx}\log A(x) \;=\; \frac{xA'(x)}{A(x)} \;=\; \frac{\sum_{k\geq 1} kA_k x^k}{\sum_{i\geq 0} A_i x^i},$$

and equate it with $x\frac{d}{dx}\log f(x) = \sum_{j\geq 1} f_j x^j$, which is computed using the laws of logarithms: exp gets cancelled, products become sums, exponents become coefficients. Finally, we write:

$$\sum_{k\geq 1} kA_k x^k \;=\; \sum_{i\geq 0, j\geq 1} A_i f_j x^{i+j},$$

so that $A_k = \frac{1}{k}\sum_{i=0}^{k-1} A_i f_{k-i} = \frac{1}{k}(A_{k-1}f_1 + A_{k-2}f_2 + \cdots + A_0 f_k)$.

a. Recall the Bell numbers $B_k = \left\{ {k \atop 1} \right\} + \left\{ {k \atop 2} \right\} + \cdots + \left\{ {k \atop k} \right\}$, which count all sets of nonempty subsets which partition $[k]$; for example $B_3 = 5$ counts $\{123\}, \{12|3\}, \{13|2\}, \{1|23\}, \{1|2|3\}$. The corresponding labeled class is constructed as: $\tilde{B} = \textsc{Set}(\textsc{Set}_{\geq 1}[1])$. Evaluate the exponential generating function $\tilde{B}(x)$, and use the Dlog Method to find a recurrence for B_k.
Extra Credit: Give a deletion transform which produces the above recurrence.

b. Apply the Dlog Method to get a recurrence for the partition numbers $p(k)$. Compare this with the recurrence from Euler's Pentagonal Number Theorem in HW 3 #2a.