Math 880 **Homework 4** **Due Fri 9/29**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. Please do not look up answers; but if you do get significant help from a reference or a person, give explicit credit.

1. Formal power series and connected graphs
We define the ring $\mathbb{C}[[x]]$ as the set of all sequences of complex numbers $(a_0, a_1, a_2, \ldots)$, which we write in the notation of power series $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, even when the series does not converge for any $x \neq 0$. Addition and multiplication are implied by the corresponding operations on power series: $(\sum_i a_i x^j)(\sum_j b_j x^j) = \sum_k c_k x^k$ with $c_k = \sum_{i+j=k} a_i b_j$; the additive identity is $0 = 0 + 0x + 0x^2 + \cdots$ and the multiplicative identity $1 = 1 + 0x + 0x^2 + \cdots$.

The ring of complex analytic functions near $x = 0$ embeds as a subring of $\mathbb{C}[[x]]$, by mapping a function to its (convergent) Taylor series. However, $\mathbb{C}[[x]]$ includes elements like the (divergent) generating function of labeled graphs, $\tilde{G}(x) = \sum_{k \geq 0} 2^{\binom{k}{2}} \frac{x^k}{k!}$.

a. Prove that the multiplication of $\mathbb{C}[[x]]$ is associative, directly from the definition (equating coefficients).

b. Now prove associativity by an analytic argument. If $A(x), B(x), C(x) \in \mathbb{C}[x]$ are *polynomials*, then $(A(x)B(x))\,C(x)$ and $A(x)\,(B(x)C(x))$ represent the same function of x, and therefore have the same Taylor series. Why does this imply the same equation for any $A(x), B(x), C(x) \in \mathbb{C}[[x]]$? *Hint:* Each coefficient of the product has contributions from only finitely many terms in the factors.

c. Prove that $A(x) = \sum_k a_k x^k$ has a reciprocal if and only if $a_0 \neq 0$. (Use either method of proof, but be rigorous.)

d. Given a list of formal series $A_n(x) = \sum_k a_k^{(n)} x^k$, we define the *formal limit* $\lim_{n\to\infty} A_n(x) = A(x)$ to mean: for each k, there is some $n(k)$ such that $a_k^{(n)} = a_k$ for all $n \geq n(k)$; i.e., the k^{th} coefficient stabilizes for large enough n. (Also called *convergence in the x-adic topology.*)

For $A(x), B(x) \in \mathbb{C}[[x]]$, we define the the composite series as:

$$A(B(x)) \;=\; \lim_{n\to\infty} a_0 + a_1 B(x) + a_2 B(x)^2 + \cdots + a_n B(x)^n.$$

PROBLEM: Show that this limit formally converges if and only if $b_0 = 0$ or if $A(x)$ is a polynomial (of finite degree).

e. Since a graph is a set of connected subgraphs, the Exponential Formula implies $\tilde{G}(x) = \exp(\tilde{C}(x))$, where $\tilde{C}(x)$ is the exponential generating function of connected graphs on k labeled vertices. Using the fact that an analytic inverse function is also a formal inverse, this implies:

$$\tilde{C}(x) \;=\; \sum_{k\geq 1} \tilde{C}_k \frac{x^k}{k!} \;=\; \log(\tilde{G}(x)) \;=\; \log\left(1 + \sum_{k\geq 1} 2^{\binom{k}{2}} \frac{x^k}{k!}\right),$$

where the outside function of the composition is the series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$. Adapt this formula to Wolfram Alpha or Mathematica to compute $\tilde{C}_k$ for $k = 4$.

Extra Credit: Verify the result by directly enumerating the connected graphs on 4 vertices, which can have 3,4,5, or 6 edges.

2a. Recall the Binomial-Theorem-type identities for Stirling cycle and partition numbers:

$$\sum_{\ell=1}^{k}(-1)^{k+\ell}\begin{bmatrix}k\\ \ell\end{bmatrix}y^{\ell} \;=\; y^{\underline{k}}, \qquad \sum_{\ell=1}^{k}\begin{Bmatrix}k\\ \ell\end{Bmatrix}y^{\underline{\ell}} \;=\; y^{k}.$$

We can interpret these as change-of-basis formulas for the $(k{+}1)$-dimensional vector space of polynomials in y of degree at most k. Explictly write out the 4×4 integer change-of-basis matrices implied by the above formulas for $k=3$, and verify that they are indeed inverse.

b. In class, we proved the first identity above using a bijective argument (cycle-to-oneline transformation on $\sum_{\ell=1}^{k}\left[\begin{smallmatrix}k\\ \ell\end{smallmatrix}\right]y^{\ell}=y^{\overline{k}}$, for $y\in\mathbb{N}$). In this problem, prove the second formula by taking $y\in\mathbb{N}$ and factoring an arbitrary mapping $f:[k]\to[y]$ as a surjective mapping $[k]\twoheadrightarrow[\ell]$ and an injective mapping $[\ell]\hookrightarrow[y]$.

3. From the labeled bigraded construction $\tilde{\mathcal{S}}=\coprod_{n\geq 0}S_n=\textsc{Set}(\textsc{Cyc}\{\tilde{1}\})$, we computed the bivariate generating function:

$$\tilde{S}(x,y) \;=\; \sum_{k,\ell\geq 0}\begin{bmatrix}k\\ \ell\end{bmatrix}y^{\ell}\frac{x^k}{k!} \;=\; \exp\!\Big(y\log\tfrac{1}{1-x}\Big) \;=\; \frac{1}{(1-x)^y}\,,$$

as well as the one-variable generating function:

$$S_k(y) \;=\; \sum_{\ell=1}^{k}\begin{bmatrix}k\\ \ell\end{bmatrix}y^{\ell} \;=\; y^{\overline{k}} \;=\; y(y{+}1)\cdots(y{+}k{-}1).$$

a. Write an equation relating $S_k(y)$ with $S_{k-1}(y)$. Deduce a Pascal-type recurrence for $\left[\begin{smallmatrix}k\\ \ell\end{smallmatrix}\right]$.

b. Consider the average or expected number of cycles in a permutation $w\in S_k$:

$$a_k \;=\; \frac{1}{k!}\sum_{w\in S_k}\mathrm{cyc}(w) \;=\; \frac{1}{k!}\sum_{\ell=1}^{k}\begin{bmatrix}k\\ \ell\end{bmatrix}\ell.$$

Show that the generating function of a_k is a partial derivative of $\tilde{S}(x,y)$:

$$A(x) \;=\; \sum_{k\geq 0}a_k x^k \;=\; \left[\frac{\partial}{\partial y}\tilde{S}(x,y)\right]_{y=1}.$$

(This is a general fact about bivariate generating functions.)

Compute $A(x)$, and conclude the asymptotic approximation $a_k\sim\log(k)$: a permutation $w\in S_k$ is expected to have about $\log(k)$ cycles. *Hint:* $\sum_{\ell=1}^{k}\frac{1}{\ell}\sim\int_1^k\frac{1}{x}dx$.