Math 880 **Homework 3** **Due Wed 9/20**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. Please do not look up answers; but if you do get significant help from a reference or a person, give explicit credit.

1. EULER PENTAGONAL NUMBER THEOREM: Let $\mathcal{Q}$ be the graded, signed class comprising all number partitions into distinct non-zero parts: $\mathcal{Q} = \{\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0),\ \ell \geq 0\}$, with $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ and $\mathrm{sgn}(\lambda) = (-1)^\ell$. Then the signed generating function $Q^{\mathrm{sgn}}(x) = \sum_{q \in \mathcal{Q}} \mathrm{sgn}(q)\, x^{|q|}$ is equal to:

$$Q^{\mathrm{sgn}}(x) \;=\; \prod_{i\geq 1}(1-x^i) \;=\; 1 + \sum_{\ell \geq 1}(-1)^\ell (x^{\ell(3\ell+1)/2} + x^{\ell(3\ell-1)/2}).$$

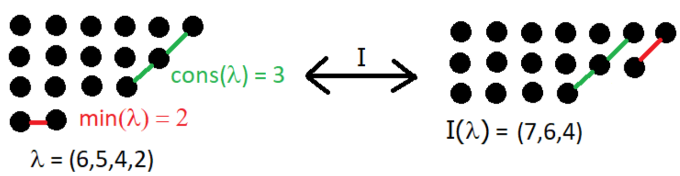
We proved this using:

INVOLUTION PRINICIPLE: Let $\mathcal{A}$ be a signed, graded class, and $I : \mathcal{A} \to \mathcal{A}$ an involutive bijection ($I^{-1} = I$) which preserves weight, $|I(a)| = |a|$, and reverses sign, $\mathrm{sgn}(I(a)) = -\mathrm{sgn}(a)$, except for fixed elements $a \in \mathcal{F} = \{a \in \mathcal{A} \text{ with } I(a) = a\}$. Then the signed generating functions of $\mathcal{A}$ and $\mathcal{F}$ are equal: $A^{\mathrm{sgn}}(x) \;=\; F^{\mathrm{sgn}}(x)$.

The desired involution on $\mathcal{A} = \mathcal{Q}$ is Franklin's bijection $I : \mathcal{Q} \to \mathcal{Q}$, which is defined in terms of the measures $\min(\lambda) = \lambda_\ell$, and:

$$\mathrm{cons}(\lambda) = \max(\{1\} \cup \{j \mid \lambda_i = \lambda_{j-1} - 1 \text{ for } i = 2, \ldots, j\}),$$

the length of the initial segment of consecutive numbers in λ. If $\min(\lambda) \leq \mathrm{cons}(\lambda)$, then I moves the bottom row of the Ferrars diagram $\lambda \in \mathcal{Q}$ to a diagonal at the top right of $I(\lambda)$. If $\min(\lambda) > \mathrm{cons}(\lambda)$, then I reverses this, moving the upper-right consecutive diagonal to form a new bottom row.



If neither operation can be perfomed on λ within $\mathcal{Q}$, we let $I(\lambda) = \lambda$.

a. Draw pictures of Ferrars diagrams paired by I for all partitions λ with $|\lambda| = 5, 6, 7$. (If you have trouble producing pictures in TeX, just give the partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$.)

b. Give a formal definition of I in set notation, and verify that it does indeed take $\mathcal{Q}$ to $\mathcal{Q}$, and is a sign-reversing, graded involution.

c. Explicitly find the two sequences of fixed elements $\lambda = I(\lambda)$, with $|\lambda| = \frac{1}{2}\ell(3\ell\pm 1)$, thereby verifying the Theorem. (For each ℓ, give two explicit partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, and show there are no others.)

2. Recall that if a complicated power series $A(x) = \sum_{k\geq 0} a_k x^k$ has an explicitly known reciprocal $B(x) = \frac{1}{A(x)} = \sum_{k\geq 0} b_k x^k$, then the equation $B(x)A(x) = 1$ gives the recurrence:

$$a_k = -\frac{1}{b_0}(b_1 a_{k-1} + b_2 a_{k-2} + \cdots + b_k a_0).$$

a. Apply the above to the partition generating function $P(x) = \sum_{k\geq 0} p(k)x^k = \prod_{i\geq 1}\frac{1}{1-x^i}$, whose reciprocal $Q(x)$ is explicitly given above. Recursively compute $p(k)$ for $k \leq 20$.

b. Give a combinatorial interpretation of $A(x)B(x) = 1$ for $A(x) = (1+x)^n$.

Extra Credit: Give a sign-reversing involution to prove the above combinatorial identity.

3. PRINCIPLE OF INCLUSION-EXCLUSION: Consider the (ungraded) combinatorial classes $\mathcal{B}_1, \ldots, \mathcal{B}_\ell \subset \mathcal{A}$. Then:

$$\#\left(\mathcal{A} - \bigcup_{j=1}^{\ell} \mathcal{B}_j\right) \;=\; \sum_{J \subset [\ell]} (-1)^{\#J}\, \#\left(\bigcap_{j \in J} \mathcal{B}_j\right).$$

a. A *derangement* is a permutation $w : [k] \xrightarrow{\sim} [k]$ with no fixed points: $w(i) \neq i$ for all i. Use PIE to find a formula for D_k, the number of derangements in S_k. Use the formula to asymptotically estimate the fraction of $w \in S_k$ which are derangements.

EXAMPLE: $D_3 = 2$, counting $w = [231]$ and $[312]$, where we write $w = [w(1)\, w(2)\, w(3)]$.

b. Prove PIE using the Involution Principle, as follows. Let $\hat{\mathcal{A}}$ be the signed class of pairs (a, J), where $a \in \mathcal{A}$, $J \subset [\ell]$, and $a \in \bigcap_{j \in J} \mathcal{B}_j$. Define $\operatorname{sgn}(a, J) = (-1)^{\#J}$. Now find a sign-reversing involution $I : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ whose fixed points are $(a, \varnothing)$ for $a \in \mathcal{A} \setminus \bigcup_{j=1}^{\ell} \mathcal{B}_j$, and show the Involution Principle reduces to PIE.

HINT: Let $I(a, J) = (a, J')$ for some J' which is one larger or smaller than J.