1b,c. Define Franklin's involution $I : \mathcal{Q} \to \mathcal{Q}$ on integer partitions with distinct parts:

$$\mathcal{Q} = \{\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0),\ \ell \geq 1\},$$

using the measure of the smallest part and of the initial string of consecutive parts:

$$m = \min(\lambda) = \lambda_\ell \quad \text{and} \quad c = \mathrm{cons}(\lambda) = \max\,\{1\} \cup \{j \leq \ell \mid \lambda_i = \lambda_{i-1}{-}1\ \forall i \leq j\};$$

$$I(\lambda_1,\ldots,\lambda_\ell) = \begin{cases} (\lambda_1{+}1,\ldots,\lambda_m{+}1,\lambda_{m+1},\ldots,\lambda_{\ell-1}) & \text{if } m \leq c \text{ but not } m = c = \ell \qquad \text{(Case 1)} \\ (\lambda_1{-}1,\ldots,\lambda_c{-}1,\lambda_{c+1},\ldots,\lambda_\ell,c) & \text{if } m > c \text{ but not } m = c{+}1 = \ell{+}1 \quad \text{(Case 2)} \\ (\lambda_1,\ldots,\lambda_\ell) & \text{if } m = c = \ell \text{ or } m = c{+}1 = \ell{+}1. \quad \text{(Case 3)} \end{cases}$$

Clearly, I preserves the size $|\lambda| = \sum_i \lambda_i$, and reverses the sign $(-1)^\ell$ in Cases 1 and 2. The fixed partitions of Case 3 are, from the definition of m and c:

$$\lambda = (2\ell{-}1, 2\ell{-}2, \ldots, \ell) \quad \text{and} \quad (2\ell, 2\ell{-}1, \ldots, \ell{+}1),$$

with respective sizes $|\lambda| = \frac{\ell}{2}(2\ell{-}1 + \ell) = \frac{1}{2}\ell(3\ell{-}1)$ and $\frac{\ell}{2}(2\ell + \ell{+}1) = \frac{1}{2}\ell(3\ell+1)$ for $\ell \geq 1$.

We check that $I(\lambda)$ is a well-defined element of $\mathcal{Q}$. Indeed, if λ belongs to Case 1, with $m \leq c \leq \ell$, the leading subsequence $(\lambda_1{+}1, \ldots,\ \lambda_m{+}1)$ would be too long to fit in λ' only if $m > \ell{-}1$, i.e. if $\ell{-}1 < m \leq c \leq \ell$, which would mean $m = c = \ell$, and λ would belong to Case 3, not Case 1.

Now consider if λ belongs to Case 2, with $m > c$. In any case, we have $\lambda_c{-}1 > \lambda_{c+1}$ by the definition of c, so that $I(\lambda) \in \mathcal{Q}$, unless the tail subsequence $(\lambda_c{-}1, \lambda_{c+1}, \ldots, \lambda_\ell, c)$ in λ' reduces to just $(\lambda_c{-}1, c)$. This happens only when $\ell < c{+}1$, meaning $\ell{-}1 < c \leq \ell$ and $c = \ell$; and the only violation would be if $\lambda_c{-}1 \leq c$, that is $m{-}1 = \lambda_\ell{-}1 = \lambda_c{-}1 \leq c < m$, which would mean $m = c{+}1 = \ell{+}1$, and λ would belong to Case 3. Thus there can be no violation, and we always have $I(\lambda) \in \mathcal{Q}$.

Claim: I is an involution, a bijection between Cases 1 and 2.

First, it is clear that the Case 1 and 2 operations are inverse to each other when they can be performed, and that neither operation can produce $I(\lambda) = \lambda'$ of Case 3 for any $\lambda \in \mathcal{Q}$.

Second, suppose λ belongs to Case 1, with its measures $m \leq c \leq \ell$, and $I(\lambda) = \lambda'$ has its own measures $\ell' = \ell{-}1$, $c' = m$. We have $m' = \lambda_{\ell-1}$ provided the tail subsequence $(\lambda_{m+1}, \ldots, \lambda_{\ell-1})$ in λ' is non-empty, that is if $m{+}1 \leq \ell{-}1$, so that $m' = \lambda_{\ell-1} > \lambda_\ell = m = c'$, and $I(\lambda) = \lambda'$ belongs to Case 2. The other possibility is $m' = \lambda_m{+}1$ if the tail is empty, that is if $m{+}1 > \ell{-}1$ and $\ell{-}2 < m \leq c \leq \ell$, so that $m = \ell{-}1$ or ℓ. If $m = \ell{-}1$, so that $m' = \lambda_{\ell-1}{+}1 > \lambda_\ell = c'$, then again $I(\lambda) = \lambda'$ belongs to Case 2. Finally, $m = \ell$ is impossible, since it would mean $\ell = m \leq c \leq \ell$ and $m = c = \ell$, so λ would belong to Case 3.

Last, suppose λ belongs to Case 2, with $m > c$, and $I(\lambda) = \lambda'$, so that $m' = c$. Then $\lambda_c{-}1 \geq \lambda_\ell{-}1 = m{-}1 \geq c \geq 1$, and it is easy to see that $c' \geq c$, so $m' = c \leq c'$, and $I(\lambda) = \lambda'$ belongs to Case 1. This concludes the proof of the Claim.