Math 880 **Homework 2** **Due Mon 9/11/17**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. Please do not look up answers; but if you do get significant help from a reference or a person, give explicit credit.

1. A shirt has 6 positions for buttons, which you sew on according to each rule below. For each problem, find the correct entry (1–12) of the Twelvefold Way table which counts the given type of button arrangements. Try to evaluate this number to solve the problem.

(a) You have a supply of 3 kinds of buttons (Black, White, Red), and you sew on any 6 of them. Example: (B,W,W,W,B,B) with R not appearing. *Ans:* Entry #1, count 3^6.

(b) Same as (a), but every color must appear. Ex: (B,W,W,W,R,B).

(c) Same as (a), but some positions might be left empty ($-$). Ex: (B,$-$,W,W,B,$-$).

(d) You have 9 distinct buttons 1,2,...,9, and you sew on any 6 of them. Ex: (8,4,5,3,1,9).

(e) The 6 positions already have 6 distinct buttons (1,2,...,6), and you snip them off and rearrange them. Ex: Rearrange to (3,1,5,6,4,2).

(f) You sew on 3 identical Black buttons, leaving 3 positions empty ($-$). Ex: (B,$-$,B,B,$-$,$-$).

(g) Same as (f), but you may sew several buttons (or none) onto each position: Ex: (B,$-$,BB,$-$,$-$,$-$).

(h) You sew on 10 Black buttons, allowing several at each position, but no positions empty. Ex: (BBB,B,BB,BB,B,B).

2. For each entry in the Twelvefold Way, define an appropriate Deletion Transform, and give the corresponding Pascal-triangle-type identity. For example, in the case of multi-choose numbers (entry 4), we saw in HW 1 #2c that $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right)$, because of the transform which removes one copy of n from M if $n \in M$, and leaves M unchanged if $n \notin M$. Use standard notation like $\binom{n}{k}$ for the entries in the Twelvefold Way, but if there is no special symbol, use $a(n,k)$.

Hints: To find the simplest recurrence, remove either a ball from the last (n^{th}) basket, or the ball with the largest label k. For the number partition entry #10, give a recurrence for $p_{\leq n}(k)$ in terms of smaller $p_{\leq n}(k)$'s, not $p_n(k)$'s; and similarly for #12.

3. Partition fallacies. Recall the number of partitions of n into at most k parts:

$$p_{\leq k}(n) := \#\left\{\lambda = (\lambda_1, \ldots, \lambda_k) \text{ such that } \lambda_1 \geq \cdots \geq \lambda_k \geq 0 \text{ and } \lambda_1 + \cdots + \lambda_k = n\right\}.$$

All of the statements below about $p_{\leq k}(n)$ are FALSE. Pinpoint the bogus step where each proof breaks down. Note that the proofs are not just incomplete or imprecise: they must be flat-out wrong, since they "prove" something false. Hint: Follow the proof on a small counterexample, and find the breakdown. Don't just give a counterexample, though: state exactly what is wrong with the reasoning.

a. FALSE: We have $p_{\leq k}(n) = \left(\!\binom{n+1}{k}\!\right)$, since each partition λ corresponds to a multiset $M := \{\lambda_k{+}1 \leq \cdots \leq \lambda_1{+}1\}$, where $1 \leq \lambda_i{+}1 \leq n{+}1$. The inverse map takes $M = \{s_1 \leq \cdots \leq s_k\}$ to $\lambda = (s_k{-}1, \ldots, s_1{-}1)$.

b. FALSE: We have $p_{\leq k}(n) = \left(\!\binom{k}{n}\!\right)$, since each partition λ corresponds to a multiset:

$$M := \{\underbrace{1, \ldots, 1}_{\lambda_1 \text{ times}}, \ldots, \underbrace{k, \ldots, k}_{\lambda_k \text{ times}}\},$$

with the total number of elements being $\lambda_1 + \cdots + \lambda_k = n$. The inverse map takes $M = \{s_1 \leq \cdots \leq s_k\}$ to $M' = \{m_1, \ldots, m_k\}$ with $m_i = \#\{j \mid s_j = i\}$, and λ is the decreasing rearrangement of M'. These two transformations undo each other, so is a bijection.

c. FALSE: We have $p_{\leq k}(n) = p_{\leq k}(n{-}1) + p_{\leq k-1}(n)$, the same recurrence as $\left(\!\binom{n}{k}\!\right)$.
Non-proof: Define a transformation T of $\lambda = (\lambda_1, \ldots, \lambda_k)$ by: $T(\lambda) := (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k{-}1)$, if $\lambda_k \geq 1$; and $T(\lambda) := (\lambda_1, \ldots, \lambda_{k-1})$ if $\lambda_k = 0$. This mapping has an obvious inverse with $T^{-1}(T(\lambda)) = \lambda$. Counting the objects on both sides gives the desired equality.