Math 880 **Homework 1: Due 9/6** **Fall 2017**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. Please do not look up answers; but if you do get significant help from a reference or a person, give explicit credit.

1. *Graded Product Principle:* Consider a graded class $\mathcal{A}$ split into subsets $\mathcal{A}_k = \{a \in \mathcal{A} \mid |a| = k\}$, with generating function $A(x) = \sum_{k\geq 0} A_k x^k$ for $A_k = \#\mathcal{A}_k$; and similarly for classes $\mathcal{B}, \mathcal{C}$. If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ with grading $|(a,b)| = |a| + |b|$, then $C(x) = A(x)B(x)$.
Hint: Show that $\mathcal{C}_k = \mathcal{A}_0 \times B_k \sqcup A_1 \times B_{k-1} \sqcup \cdots \sqcup A_k \times B_0$, apply the ordinary (ungraded) Product and Sum Principles, and compare to $\sum_{i\geq 0} A_i x^i \cdot \sum_{j\geq 0} B_j x^j = \sum_{i,j\geq 0} A_i B_j x^{i+j}$.

2. Define a *multiset* to be like a set, but with repeat elements allowed, such as $M = \{1,3,3,5\}$. The number $\left(\!\binom{n}{k}\!\right)$, n multi-choose k, counts all multisets of k elements chosen from $[n] = \{1,2,\ldots,n\}$. EXAMPLE: $\left(\!\binom{2}{3}\!\right) = 4$ counts ways to stock an ice bucket with 3 cans of soda of 2 possible kinds: $M = \{1,1,1\}, \{1,1,2\}, \{1,2,2\}$, or $\{2,2,2\}$. Analyze $\left(\!\binom{n}{k}\!\right)$ similarly to $\binom{n}{k}$.

a. Generalize the Multiplicity Transform from sets to multisets. That is, transform the data of a multiset M into a list $(m_1,\ldots,m_n)$, with $m_i \geq 0$ being the number of times i appears in M, with the grading function $|M| = \sum_i m_i$.

For fixed n, use this to define a graded bijection between the class $\mathcal{M}$ of all multisets with elements in $[n]$, and the graded product class $\mathbb{N}^n$, where $\mathbb{N} = \{0,1,2,\ldots\}$ and $|i| = i$.

b. Use the Graded Product Principle to determine the generating function $M(x) = \sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) x^k$, and use the Taylor Coeffiicient Formula to find $\left(\!\binom{n}{k}\!\right)$ explicitly.

c. Use a Deletion Transform to find a Pascal Triangle type recursion for $\left(\!\binom{n}{k}\!\right)$.

d. The Accordion Transform changes multisets to sets: $M = \{t_1 \leq \cdots \leq t_k\}$ with $t_i \in [n]$ corresponds to $S = \{s_1 < \cdots < s_k\}$ with $s_i = t_i + i - 1$. Show this gives a bijection between appropriate classes, and conclude that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$.

3. Consider the recurrence: $A_n = A_{n-1} + 2A_{n-2}$ for $n \geq 2$ with initial values $A_0 = A_1 = 1$, beginning $1,1,3,5,11,21,\ldots$. Analyze this recurrence similarly to the Fibonacci sequence.

a. Find a simple formula for the generating function $A(x) = \sum_{n\geq 0} A_n x^n$.

b. Find a partial fraction expansion of $A(x)$ to get an explicit formula for A_n.

c. Pick the dominant term $\frac{c}{1-x/\lambda}$ in the partial fraction decomposition, corresponding to the singularity $x = \lambda$ nearest the origin, and show the asymptotic approximation $A_n \sim c\lambda^{-n}$. (By definition, this means $\lim_{n\to\infty} \frac{A_n}{c\lambda^{-n}} = 1$, i.e. the percentage error goes to zero.)

d. Expand $A(x) = \sum_{\ell\geq 0} (x + 2x^2)^\ell$, and reverse the Graded Product and Sum Principles to find a class $\mathcal{A}_n$ of combinatorial objects counted by A_n. *Hint:* Consider a "marked" number $2'$, which is distinct from ordinary number 2, but adds just like it.