Math 482 — **The Twelvefold Way** — **Spring 2014**

We count the number of possible functions f with input set $[k] = \{1, 2, \ldots, k\}$ and output set $[n] = \{1, 2, \ldots, n\}$, subject to restrictions (injective or surjective). We may picture f as a way of distributing k balls (marked $1, \ldots, k$) into n baskets (marked $1, \ldots, n$). A map is injective if each basket contains at most one ball, or surjective if no basket is empty.

Indistinguishable $[k]$ means we consider two functions the same whenever they differ by a permutation of the inputs $[k]$; so we picture the k balls as identical, unmarked. Similarly, *indistinguishable* $[n]$ means we consider classes of functions up to permutation of the outputs $[n]$, so we picture the n baskets as identical and movable, and we cannot distinguish a first basket, second basket, etc.

$f : [k] \to [n]$		ALL FUNCTIONS	INJECTIONS	SURJECTIONS
DIST	DIST	① n^k $n^k = n \cdot n^{k-1}$	② $n^{\underline{k}}$ $n^{\underline{k}} = (n-k+1)\, n^{\underline{k-1}}$	③ $\operatorname{surj}(k,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$ $\operatorname{surj}(k,n) = n \operatorname{surj}(k-1, n-1) + n \operatorname{surj}(k-1, n)$
IND	DIST	④ $\left(\!\binom{n}{k}\!\right) = \frac{n^{\overline{k}}}{k!}$ $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right)$	⑤ $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$ $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$	⑥ $\left(\!\binom{n}{k-n}\!\right) = \binom{k-1}{n-1}$
DIST	IND	⑦ $\left\{{k \atop 1}\right\} + \left\{{k \atop 2}\right\} + \cdots + \left\{{k \atop n}\right\}$	⑧ $\left[\begin{array}{ll} 1 & \text{if } k \le n \\ 0 & \text{otherwise} \end{array}\right.$	⑨ $\left\{{k \atop n}\right\} = \frac{\operatorname{surj}(k,n)}{n!}$ $\left\{{k \atop n}\right\} = \left\{{k-1 \atop n-1}\right\} + n \left\{{k-1 \atop n}\right\}$
IND	IND	⑩ $p_{\le n}(k) = p_1(k) + p_2(k) + \cdots + p_n(k)$	⑪ $\left[\begin{array}{ll} 1 & \text{if } k \le n \\ 0 & \text{otherwise} \end{array}\right.$	⑫ $p_n(k)$ $p_n(k) = p_{n-1}(k-1) + p_n(k-n)$

- Binomial coefficient or choose-number $\binom{n}{k}$. Multiset number or multi-choose number $\left(\!\binom{n}{k}\!\right)$. Stirling partition number (second kind) $\left\{{k \atop n}\right\}$.
- Stirling cycle number (first kind) $\left[{n \atop k}\right]$ counts permutations of n having k cycles. Recurrence: $\left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right]$.
- Bell number $B(k) = \left\{{k \atop 1}\right\} + \left\{{k \atop 2}\right\} + \cdots + \left\{{k \atop k}\right\}$. Ordered Bell number $R(k) = \operatorname{surj}(k,1) + \operatorname{surj}(k,2) + \cdots + \operatorname{surj}(k,k)$.
- partition number $p(k) = p_{\le k}(k) = p_1(k) + p_2(k) + \cdots + p_k(k)$.
- Fibonacci number $F_k = F_{k-1} + F_{k-2}$ from $F_0 = 0$, $F_1 = 1$; formula $F_k = \frac{1}{\sqrt{5}}(\phi^k - (-\psi)^k)$, where $\phi = \frac{\sqrt{5}+1}{2}$, $\psi = \frac{\sqrt{5}-1}{2}$.
- Catalan number $C_k = \sum_{i=0}^{k-1} C_i C_{k-i}$ from $C_0 = 1$; formula $C_k = \frac{1}{k+1}\binom{2k}{k}$.
- Derangement number (permutations without fixed points) $D_k = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!})$
- Number of Cayley (labeled) trees: $T_n = n^{n-2}$. Number of unlabeled trees: $t_n = ??$