Math 482 **Homework 4/4 Solutions** **Spring 2014**

Homework: `math.msu.edu/~magyar/Math482/Old.htm#4-4`.

1. Let G be a polyhedral graph with n vertices, q edges, r regions.

   Claim (a): $q \leq 3r-6$. *Pf:* By Edge-Vertex, $3n \leq \sum_v \deg(v) = 2q$. By Euler, this means $3(2-r+q) \leq 2q$, which rearranges to the desired formula.

   Claim (b): $n \leq 2r-4$. *Pf:* As before, $3n \leq 2q = 2(n+r-2)$, rearranging to the Claim.

   Claim (c): There is some region with $\deg(R) \leq 5$.
   Pf: The average degree of the regions is:

$$A_r \;=\; \tfrac{1}{r}\sum_R \deg(R) = \frac{2q}{r} \;\leq\; \frac{2(3r-6)}{r} \;=\; 6 - \tfrac{12}{r} < 6.$$

   Thus there is some R with $\deg(R) \leq A_R < 6$, meaing $\deg(R) \leq 5$.

2a. The potential energy of $F(x) = -x$ is: $PE(x_1) = -\int_0^{x_1}(-x)\,dx = \frac{1}{2}x_1^2$.

2b. For $F(x) = -(x-a_1) - (x-a_2) = -2x + a_1 + a_2$, we have:

$$PE(x) \;=\; x^2 - (a_1+a_2)x.$$

Completing the square gives: $PE(x) = (x-c)^2 - e$, where $c = \frac{1}{2}(a_1+a_2)$ and $e = c^2$. This is clearly a parabola having a unique minimum.

Also, an equilibrium point $x = x_0$ means $0 = F(x_0) = -PE'(x_0)$ since $-PE(x)$ is the anti-derivative of $F(x)$. Thus, each equilibrium point of $F(x)$ is a critical point of $PE(x)$; but $PE(x)$ has only one critical point (its minimum $x = c$).

2c. The same reasoning applies for $F(x) = -(x-a_1) - \cdots - (x-a_n)$ as for $n = 2$ above.

3a. We compute the line integral of $\vec{F}(x, y) = -(x, y)$ over $\vec{r}(t) = (tx_1, ty_1)$ for constants (x_1, y_1) and $0 \leq t \leq 1$:

$$\begin{aligned} PE(x_1, y_1) \;&=\; -\oint \vec{F}(\vec{r}) \cdot d\vec{r} \;=\; -\int_0^1 \vec{F}(x(t), y(t)) \cdot \vec{r}'(t)\,dt \\ &=\; \int_0^1 (tx_1, ty_1) \cdot (x_1, y_1)\,dt \;=\; \int_0^1 (x_1^2 + y_1^2)t\,dt \;=\; \tfrac{1}{2}(x_1^2 + y_1^2). \end{aligned}$$

This is an upward-curving paraboloid.

3c. Let us examine the case of two Hooke forces: $\vec{F}(x, y) = -((x, y)-(a_1, b_1))-((x, y)-(a_2, b_2)) = -2(x, y) + (a_1+a_2, b_1+b_2)$. This integrates as before to:

$$\begin{aligned} PE(x, y) \;&=\; x^2 + y^2 + (a_1+a_2)x + (b_1+b_2)x \\ &=\; (x-c)^2 + (y-d)^2 - e, \end{aligned}$$

where we have completed the square: $c = \frac{1}{2}(a_1 + a_2)$, $d = \frac{1}{2}(b_1 + b_2)$, $e = c^2 + d^2$. This is once again a paraboloid with a unique minimum $(x_0, y_0) = (c, d)$.

The minimum point is a critcal point, where:

$$\operatorname{grad} PE(x_0, y_0) \;=\; \nabla PE(x_0, y_0) \quad = \left(\tfrac{\partial}{\partial x}PE(x_0, y_0),\, \tfrac{\partial}{\partial y}PE(x_0, y_0)\right) \;=\; (0, 0).$$

But, by the Fundamental Theorem of Calculus for line integrals (or by direct computation), we have $\nabla PE(x, y) = \vec{F}(x, y)$, so that again, the critical points of $PE(x, y)$ are precisely the equilibrium points where $\vec{F}(x, y) = (0, 0)$; but $PE(x, y)$ has only one critical point, its minimum.