Math 482 **Homework 4/18 Solutions** **Spring 2014**

Homework: math.msu.edu/~magyar/Math482/Old.htm#4-18.

1a. The counter-clockwise 90° rotation mapping $R:\mathbb{R}^2\to\mathbb{R}^2$ takes the x-axis $(1,0)$ to the y-axis $(0,1)$, and the y-axis $(0,1)$ to $(-1,0)$. Since it is linear, we can thus compute $R(a_1,a_2)$ as: $\begin{bmatrix}0&-1\\1&0\end{bmatrix}\cdot\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}-a_2\\a_1\end{bmatrix}$; meaning, $(a_1,a_2)^\perp=R(a_1,a_2)=(-a_2,a_1)$.

1b. For $a=(a_1,a_2)$ and $b=(b_1,b_2)$, we have:

$$a^\perp\!\cdot b \;=\; (-a_2,a_1)\cdot(b_1,b_2)=-a_2b_1+a_1b_2=\det(a,b).$$

Thus, $\{a,b\}$ is a right-handed coordinate system $\iff$ $a^\perp\!\cdot b>0$ $\iff$ $\det(a,b)>0$.

1c. We have:

$$\begin{aligned}(a\times b)\cdot c &= (a_2b_3-a_3b_2,-a_1b_3+a_3b_1,a_1b_2-a_2b_1)\cdot(c_1,c_2,c_3)\\ &= (a_2b_3-a_3b_2)c_1-(a_1b_3-a_3b_1)c_2+(a_1b_2-a_2b_1)c_3\\ &= \det(a,b,c),\end{aligned}$$

using the third-row minor expansion of the determinant.

1d. In the plane, the left side of the directed line $\vec{uv}$ is the side of the perpendicular $(v-u)^\perp$. Thus, w is on the left side of $\vec{uv}$ when $w-u$ makes an acute angle with $(v-u)^\perp$:

$$(v-u)^\perp\cdot(w-u) \;>\; 0$$

To determine the orientation (handedness) of the basis $\{(w,1),(v,1),(u,1)\}$, we evaluate the determinant:

$$\begin{aligned}\det\begin{bmatrix}w&1\\v&1\\u&1\end{bmatrix} &= \det\begin{bmatrix}w-u&0\\v-u&0\\u&1\end{bmatrix} && \text{since } \det(a,b,c)=\det(a-c,b,c)=\det(a,b-c,c)\\ &= \det\begin{bmatrix}w-u\\v-u\end{bmatrix} && \text{by minor expansion}\\ &= -\det\begin{bmatrix}v-u\\w-u\end{bmatrix} && \text{because } \det(a,b)=-\det(b,a)\\ &= -(v-u)^\perp\cdot(w-u), && \text{by Prob. 1b,}\end{aligned}$$

and the last expression is negative by hypothesis.

Another way to express the Lemma is: if $\{v-u,w-u\}$ is a right-handed coordinate system in $\mathbb{R}^2$, then $\{(w,1),(v,1),(u,1)\}$ is a left-handed coordinate system in $\mathbb{R}^3$.

2. By Solutions 4/16 #1c, the top triangle of the polyhedron P has corners:

$$v_4'=(1,1,0),\quad v_5'=(2,0,1),\quad v_6'=(0,0,0).$$

A normal vector to this plane is:

$$\vec{n}=(v_4'-v_6')\times(v_5'-v_6')=(1,1,0)\times(2,0,1)=(1,-1,-2).$$

The $\vec{n}$-height of each corner is: $\vec{n}\cdot v_4'=\vec{n}\cdot v_5'=\vec{n}\cdot v_6'=0$, so the equation of the plane is:

$$\vec{n}\cdot v'=x-y-2z=0.$$

In fact, the points $(x,y,z)\in P$ satisfy $x-y-2z\geq 0$, or equivalently $-x+y+2z\leq 0$