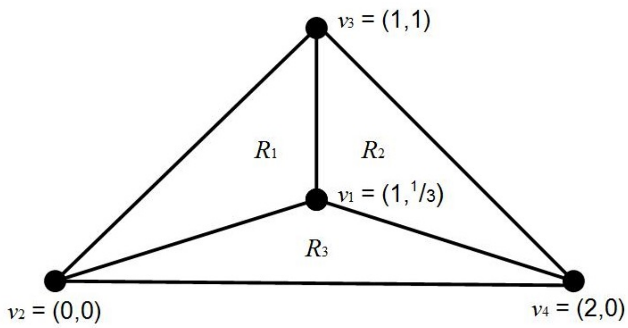
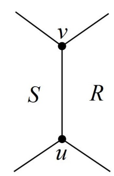
Homework: math.msu.edu/∼magyar/Math482/Old.htm#4-14.

1a,b. We work with the equilibrium stressed graph:



Starting with $q_1 = (0,0,0)$, we use the recursive formula based on the paradigm of neighboring regions:



In our case, we have the left region $S = R_1$, the right region $R = R_2$, the top vertex $v = v_3$, and the bottom vertex $u = v_1$, so that:

$$\begin{aligned} q_2 &= q_R = q_S + (v,1)\times(u,1) \\ &= q_1 + (v_3,1)\times(v_1,1) = (0,0,0) + (1,1,1)\times(1,\tfrac{1}{3},1) \\ &= \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \\ 1 & \frac{1}{3} & 1 \end{bmatrix} = (1-\tfrac{1}{3},\ -0,\ \tfrac{1}{3}-1) = (\tfrac{2}{3},0,-\tfrac{2}{3}) \end{aligned}$$

To get q_3, we use the recursive formula with $S = R_2$ at left and $R = R_3$ at right:

$$q_3 = q_2 + (v_4,1)\times(v_1,1) = (\tfrac{2}{3},0,-\tfrac{2}{3}) + (2,0,1)\times(1,\tfrac{1}{3},1) = (\tfrac{1}{3},-1,0).$$

On the other hand, we could get q_3 by tilting our view to see $S = R_1$ at left, $R = R_3$ at right, $v = v_1$ at top, $u = v_2$ at bottom:

$$q_3 = q_1 + (v_1,1)\times(v_2,1) = (0,0,0) + (1,\tfrac{1}{3},1)\times(0,0,1) = (\tfrac{1}{3},-1,0).$$

It is no coincidence that we get the same q_3 both ways.

LEMMA: Let G be any equilibrium stressed graph, with a vertex u having neighbors $v_1,\ldots,v_d$ and surrounding regions $R_1,\ldots,R_d$. If we apply the recursive rule to q_1 to successively obtain $q_2,\ldots,q_d,q_1'$, then $q_1 = q_1'$.

Proof: Let us assume that, looking from u along the edge uv_i, we have R_i on the left and R_{i+1} on the right. Then $q_{i+1} \;=\; q_i \,+\, (v_i, 1){\times}(u, 1)$, and iterating this all the way around u gives:

$$q_1' \;=\; q_1 \,+\, \textstyle\sum_{i=1}^{d}(v_i, 1){\times}(u, 1) \;=\; q_1 \,+\, (\textstyle\sum_{i=1}^{d} v_i, d){\times}(u, 1).$$

Now, the vertices are in equilibrium position, so $\sum_{i=1}^{d}(v_i{-}u) = 0$, i.e. $\sum_{i=1}^{d} v_i = du$. Therefore:

$$q_1' \;=\; q_1 \,+\, (du, d){\times}(u, 1) \;=\; q_1 \,+\, d\,(u, 1){\times}(u, 1) \;=\; q_1\,,$$

since $(u, 1) \times (u, 1) = 0$.

1c. The dot product of two vectors in the xy-plane is always vertical. The effect of raising the v_i positions to $(v_i, 1)$ is to tilt the q_j's, making them independent directions, perpedicular to independent face-planes of P.

1d. Now, v_1 is a corner of R_1, R_2, R_3, so we may compute:

$$\begin{aligned} h_1 &= q_1 \cdot (v_1, 1) = (0, 0, 0) \cdot (1, \tfrac{1}{3}, 1) = 0 \\ &= q_2 \cdot (v_1, 1) = (\tfrac{2}{3}, 0, -\tfrac{2}{3}) \cdot (1, \tfrac{1}{3}, 1) = 0 \\ &= q_3 \cdot (v_1, 1) = (\tfrac{1}{3}, -1, 0) \cdot (1, \tfrac{1}{3}, 1) = 0. \end{aligned}$$

Also, v_2 is a corner of R_1, R_3, so: $h_2 = q_1 \cdot (v_2, 1) = (\tfrac{1}{3}, -1, 0) \cdot (0, 0, 1) = 0$, or alternatively $h_2 = q_3 \cdot (v_2, 1) = (\tfrac{1}{3}, -1, 0) \cdot (0, 0, 1) = 0$; and similarly $h_3 = q_2 \cdot (v_3, 1) = 0$. This means that the polyhedron P has a triangular face corresponding to region R_1, with vertices:

$$(v_1, h_1) = (1, \tfrac{1}{3}, 0),\ (v_2, h_2) = (0, 0, 0),\ (v_3, h_3) = (1, 1, 0).$$

That is, this face is horizontal at level 0. The final vertex is: $h_4 = q_2 \cdot (v_4, 1) = (\tfrac{2}{3}, 0, -\tfrac{2}{3}) \cdot (2, 0, 1) = \tfrac{2}{3}$, so the final vertex is: $(v_4, h_4) = (2, 0, \tfrac{2}{3})$. The faces of P are defined as the triangles between the given vertices. The faces above the interior regions R_1, R_2, R_3 are normal to the corresponding vectors q_j, with an extra triangle on top corresponding to the external triangle v_2, v_3, v_4.