Homework: math.msu.edu/∼magyar/Math482/Old.htm#3-26.

1a. Proposition: Let $P$ be a polyhedron with $n$ vertices, $q$ edges, $r$ faces. Then $P$ is triangular if and only if $q = 3n - 6$ and $r = 2n - 4$.

*Proof.* ($\Longrightarrow$): Suppose $P$ is triangular. Then its edge-graph is a maximal planar graph with all region-degrees $\deg(R) = 3$. Thus $3r = \sum_R \deg(R) = 2q$. Substituting $r = 2 - n + q$ yields $q = 3n - 6$, whereas substituting $q = n + r - 2$ yields $r = 2n - 4$.

($\Longleftarrow$): Suppose $q = 3n - 6$, but some face of $P$ has at least 4 edges. Then this face corresponds to a region in the edge-graph $G$ which can be traversed by an extra edge $e$, producing a planar graph $G' = G + e$ with $n$ vertices, $q' = q + 1 = 3n - 5$ edges.

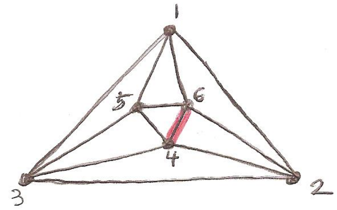
Since $G'$ has all $\deg(R) \geq 3$, and all edges lie between 2 distinct regions, we know: $3r' \leq \sum_R \deg(R) = 2q'$. Substituting $r' = 2 - n + q'$, we find $q' \leq 3n - 6$. This contradicts $q' = 3n - 5$, so the supposition must have been false. That is, $P$ can have only triangular faces.

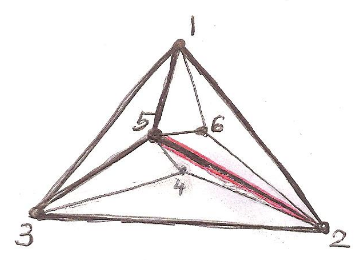
We can make a similar argument assuming $r = 2n - 4$.

1b. The octahedron is a standard polyhedron with $(n, r) = (6, 8)$. We can modify this without changing $(n, r)$: take a pair of adjacent triangles forming a quadrilateral with diagonal, and flip over the diagonal to the opposite diagonal. This leaves $(n, q, r)$ the same, but it changes the vertex degrees, making a different graph.

For example, the octahedron can be constructed as an anti-prism on the vertices $V = [6]$, giving $G$ with two 3-cycles $1, 2, 3$ and $4, 5, 6$, and connecting edges $15, 16, 24, 26, 34, 35$, giving all $\deg(v) = 4$ (see the figure below left). If we replace edge 46 (marked in red) with the opposite diagonal 52 (marked in red below right), the resulting $G'$ has vertex degrees $3, 3, 4, 4, 5, 5$; so the two cannot be isomorphic as graphs, much less give combinatorially equivalent polyhedra.

It is not very clear how to lift $G'$ to a polyhedron, but Steinitz's Theorem says it is possible. In fact, if we remove vertices 4 and 6, we are left with a tetrahedron graph (the cycle 1,2,3 with 5 in the center, marked with dark lines below right). The original $G'$ is obtained by stacking a shallow pyramid with peak 4 onto the face 235, and another shallow pyramid with peak 6 onto the face 125.




2a,b,c. The pyramid over a hexagon has $(n, r) = (7, 7)$ has vertex degrees $3, 3, 3, 3, 3, 3, 6$. The graph $G$ with $V = [7]$ and $E = \{12, 23, 34, 45, 51, 67, 61, 62, 63, 71, 74, 75\}$ has vertex degrees $3, 3, 3, 3, 4, 4, 4$, clearly making it different from the pyramid graph. To draw a polyhedron corresponding to $G$, draw a pentagonal base, the cycle 12345, with a ridge-edge 67 drawn above the line (not an edge) between vertices 2 and 5. Then there are triangular slopes descending to all the vertices 1,2,3,4,5, and a trapezoid slope between the ridge 67 and the parallel edge 34.

2d. Peform the same kind of diagonal-flip as in 1(b), replacing 16 with the opposite diagonal 27, obtaining $G'$ with vertices $V' = [7]$, consisting of a 5-cycle 1,2,3,4,5, an edge 67 inside it, and extra edges from 6 to $2, 3$ and from 7 to $1, 4, 5$

This one is pretty hard to visualize. It can be obtained from the polyhedron in 2(c) by tilting the ridge line 67, within the same plane as the bottom edge 34.

3a. The description of the edges of the cyclic polyhedron $\mathrm{Cyc}(k)$ can be seen for $k = 6$ from the second picture: just label the vertices $v_1, \ldots, v_6$, left-to-right.

3b. The description shows $q = n + 2(n-3) = 3n - 6$ and $r = 2(n-2) = 2n - 4$. By 1(a), this is the same as any other triangular polyhedron (with maximal planar edge-graph) on $n$ vertices.

3c. Draw $v_1 = (1, 0)$, $v_2 = (-1, 0)$, $v_i = (0, i-2)$ for $i = 3, \ldots, n$. Then the edges connect $v_1v_2$, and (for $i \geq 3$): $v_iv_{i+1}$, $v_1v_i$ , and $v_2v_i$.

4a. The only graph with $(n, r) = (4, 4)$ is the complete graph $K_4$, so all polyhedra are combinatorially equivalent.

4b. The unique polyhedron with $(n, r) = (6, 5)$ is the triangular prism, the vertex-truncation of a tetrahedron. The dual with $(n, r) = (5, 6)$ is the triangular bipyramid, which is a tetrahedron with another tetrahedron stacked onto a face.

5. For all the above graphs, removing any two vertices leaves the remaining vertices connected.