Math 482 **Homework 3/10 Solutions** **Spring 2014**

Homework: math.msu.edu/~magyar/Math482/Old.htm#3-10.

1a. For values of r_n, see the Online Encyclopedia of Integer Sequences #A000081:

n	0	1	2	3	4	5	6	7	8	9	10
r_n	0	1	1	2	4	9	20	48	115	286	719

The recurrence for $n{+}1 = 10$ has terms corresponding to $k = 1, \dots, 9$ and $i|k$:

$$(i,k) = (1,1), (2,2), (1,2), (3,3), (1,3), (4,4), (2,4), (1,4), (5,5), (1,5), (6,6), (3,6), (2,6), (1,6),$$
$$(7,7), (1,7), (8,8), (4,8), (2,8), (1,8), (9,9), (3,9), (1,9),$$

corresponding to the formula:

$$\begin{aligned} r_{10} \;=\; & \tfrac{1}{9}(\,(1r_1)r_9 + (2r_2+1r_1)r_8 + (3r_3+1r_1)r_7 + (4r_4+2r_2+1r_1)r_6 + (5r_5+1r_1)r_5 \\ & + (6r_6+3r_3+2r_2+1r_1)r_4 + (7r_7+1r_1)r_3 + (8r_8+4r_4r_4+2r_2+1r_1)r_2 + (9r_9+3r_3+1r_1)r_1\,). \end{aligned}$$

1b. The transformation is surjective, but not injective. In fact, whenever an unlabeled rooted tree has non-trivial symmetries (mappings onto itself taking vertices to vertices, root to root, edges to edges), then any two labelings which are symmetric to each other count as the same labeled tree.

For example, the tree: $T = \bullet - \odot - \bullet$ has the trivial symmetry and a reflection symmetry, so the two symmetric labelings are the same tree: $L = 1 - 2 - 3 = 3 - 2 - 1$. (These define the same vertex and edge sets.)

Hence we always have more pairs (T, σ) than distinct labeled trees L, and $r_n n! > n^{n-1}$; except for the trivial cases $n = 1, 2$ for which there are no symmetric trees.

2a. See http://www.math.msu.edu/~magyar/Math482/Old.html#2-12 .

2b. Starting with $\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1} \frac{1}{1-x^i}$, we apply $xD\log$ to get:

$$\begin{aligned} \frac{\sum_{n\geq 1} np(n)x^n}{\sum_{m\geq 0} p(m)x^m} &= \sum_{i\geq 1} xD\log\frac{1}{1-x^i} \;=\; \sum_{i\geq 1}\frac{ix^i}{1-x^i} \\ &= \sum_{i\geq 1}\sum_{j\geq 1} ix^{ij} \;=\; \sum_{k\geq 1}\sum_{i|k} ix^k \;=\; \sum_{k\geq 1}\mathrm{sdiv}(k)x^k, \end{aligned}$$

where we define $\mathrm{sdiv}(k) = \sum_{i|k} i$, the sum of all whole-number divisors of k. Clearing denominators gives:

$$\sum_{n\geq 1} np(n)x^n \;=\; \sum_{k\geq 1}\mathrm{sdiv}(k)x^k \times \sum_{m\geq 0} p(m)x^m.$$

Multiplying out the right side and substituting $n = k + m$, the coefficient of x^n is a convolution, and we get:

$$p(n) \;=\; \frac{1}{n}\sum_{k=1}^{n}\mathrm{sdiv}(k)\,p(n{-}k).$$

For example:

$$\begin{aligned} p(4) &= \tfrac{1}{4}(\mathrm{sdiv}(1)\,p(3) + \mathrm{sdiv}(2)\,p(2) + \mathrm{sdiv}(3)\,p(1) + \mathrm{sdiv}(4)\,p(0)) \\ &= \tfrac{1}{4}((1)(3) + (3)(2) + (4)(1) + (7)(1)) = 5 \end{aligned}$$

This use all $p(n{-}k)$ for $k = 1, \ldots, n$, whereas Euler's recurrence uses only $p(n{-}k)$ where k is a pentagonal number.

3a. We write:

$$x + x^2 + r_3x^3 + r_4x^4 + \cdots \;=\; x\,\frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{(1-x^3)^{r_3}}\,\frac{1}{(1-x^4)^{r_4}}\cdots$$

$$=\; x\,(1{+}x{+}x^2{+}x^3{+}x^4{+}\cdots)(1{+}x^2{+}x^4{+}\cdots)(1+\left(\!\binom{r_3}{1}\!\right)x^3+\cdots)(1+\left(\!\binom{r_4}{1}\!\right)x^4+\cdots)\cdots.$$

The x^3 terms are: $x{\cdot}x^2{\cdot}1{\cdot}1 + x{\cdot}x^2{\cdot}1{\cdot}1$, so $r_3 = 2$. The x^4 terms are: $x{\cdot}x^3{\cdot}1{\cdot}1 + x{\cdot}x^3{\cdot}1{\cdot}1 + x{\cdot}1{\cdot}1{\cdot}\left(\!\binom{r_3}{1}\!\right)x^3$ so $r_4 = 2 + \left(\!\binom{r_3}{1}\!\right) = 2 + r_3 = 4$.

3b. Expanding the equation gives:

$$\sum_{n\geq 0} r_{n+1}x^n \;=\; \prod_{i\geq 1}\frac{1}{(1-x^i)^{r_i}} \;=\; \prod_{i\geq 1}\sum_{k_i\geq 0}\left(\!\binom{r_i}{k_i}\!\right)x^{ik_i}.$$

Clearly, the x^n terms on the right are of the form $\left(\!\binom{r_1}{k_1}\!\right)x^{1k_1}{\cdot}\left(\!\binom{r_2}{k_2}\!\right)x^{2k_2}{\cdot}\left(\!\binom{r_3}{k_3}\!\right)x^{3k_3}\cdots$, where the indices $(k_1, k_2, k_3, \ldots)$ are chosen so that $1k_1 + 2k_2 + 3k_3 + \cdots = n$. Thus, we get the awful recurrence:

$$r_{n+1} \;=\; \sum_{\substack{k_1,k_2,k_3\ldots\,\geq 0\\ 1k_1+2k_2+3k_3+\cdots=n}}\;\prod_{i\geq 1}\left(\!\binom{r_i}{k_i}\!\right) \;=\; \sum_{\substack{k_1,k_2,k_3\ldots\,\geq 0\\ 1k_1+2k_2+3k_3+\cdots=n}}\left(\!\binom{r_1}{k_1}\!\right)\left(\!\binom{r_2}{k_2}\!\right)\left(\!\binom{r_3}{k_3}\!\right)\cdots.$$

See the example in #3(c). This recurrence uses a huge number of terms; in fact, the $(k_1, k_2, k_3, \ldots)$ indices are a transformed version of integer partitions of n, and the number of terms on the right side is precisely $p(n)$.