Homework: math.msu.edu/∼magyar/Math482/Old.htm#2-26.

1-1. In HW 2/24 #1, we considered a_n, the number of unlabeled ordered trees with n vertices, whose ordinary generating function $f(x)$ satisfies the recurrence equations:

$$f(x) - f(x)^2 = x\,, \qquad f(x) = x/(1{-}f(x))\,.$$

In fact, $f(x)$ is the inverse function of $g(x) = x - x^2$.

Now, the first version of Lagrange Inversion tells us that:

$$a_n = \tfrac{1}{n}\,[x^{-1}]\,\frac{1}{g(x)^n} = \tfrac{1}{n}[x^{-1}]\,\frac{1}{(x{-}x^2)^n} = \tfrac{1}{n}[x^{-1}]\,\frac{1}{x^n(1{-}x)^n}$$

That is, we must determine the x^{-1} term of the Laurent series $x^{-n}(1{-}x)^{-n}$. This is clearly the x^{n-1} term of $(1{-}x)^{-n}$, which is a negative binomial from the Known Series: $(1{-}x)^{-n} = \sum_{j\geq 0}(\!(\binom{n}{j})\!)x^j$. We conclude that:

$$a_n = \tfrac{1}{n}(\!(\tbinom{n}{n-1})\!) = \tfrac{1}{n}\tbinom{2n-2}{n-1}.$$

We could also obtain this more quickly from the second version of Lagrange inversion. Given $f(x) = x\,\phi(f(x))$, with $\phi(x) = \frac{1}{1-x}$, we have:

$$a_n = \tfrac{1}{n}\,[x^{n-1}]\,\phi(x)^n = \tfrac{1}{n}\,[x^{n-1}]\,\tfrac{1}{(1-x)^n} = \tfrac{1}{n}(\!(\tbinom{n}{n-1})\!) = \tfrac{1}{n}\tbinom{2n-2}{n-1}.$$

Incidentally, this means $C_n = a_{n+1} = \frac{1}{n+1}\binom{2n}{n}$, which we obtained with some difficultiy from the explicit formula for the generating function (HW 1/24 #4).

1-2 For unlabeled ordered binary trees, the generating function recurrence equaton is: $f(x) = x(1 + f(x)^2)$, i.e. $f(x) = x\,\phi(f(x))$ for $\phi(x) = 1 + x^2$. By the second version of Lagrange Inversion:

$$a_n = \tfrac{1}{n}\,[x^{n-1}]\,\phi(x)^n = \tfrac{1}{n}\,[x^{n-1}]\,(1{+}x^2)^n = \tfrac{1}{n}\tbinom{n}{\frac{1}{2}(n-1)},$$

where we define $\binom{n}{j} = 0$ if j is not a whole number. That is, $a_{2k+1} = \frac{1}{2k+1}\binom{2k+1}{k}$. This simplifies to $a_{2k+1} = C_k$ (Solutions 2/24 #2a).

1-3 For unlabeled ordered ternary trees, the generating function recurrence equation is: $f(x) = x\,\phi(f(x))$ with $\phi(x) = 1 + x^3$. Just as in Prob 1-2, we get: $a_n = \frac{1}{n}\binom{n}{\frac{1}{3}(n-1)}$. That is, $a_{3k+1} = \frac{1}{3k+1}\binom{3k+1}{k} = \frac{1}{2k+1}\binom{3k}{k}$, which is a new analog of Catalan numbers.

1-4 For labeled rooted trees, the exponential generating function recurrence equation is: $\tilde{f}(x) = x\,e^{f(x)} = x\,\phi(f(x))$ with $\phi(x) = e^x$, so Lagrange Inversion gives:

$$\frac{a_n}{n!} = \frac{1}{n}\,[x^{n-1}]\,\phi(x)^n = \frac{1}{n}\,[x^{n-1}]\,e^{nx} = \frac{1}{n}\,\frac{n^{n-1}}{(n{-}1)!}.$$

That is, $a_n = n^{n-1}$, gives another proof of Cayley's Theorem (considering HW 2/24 #4a).

2a. Let a_n be the number of conjugal trees, meaning unlabeled ordered trees in which each node has weight 1 (a single) or weight 2 (a couple). Recursive choice algorithm: a conjugal family tree is either a single or couple ancestor node, and a list of: no children or child trees (T_1) or (T_1, T_2) or $\ldots$.

2b. Since these are unlabeled trees, we use an ordinary generating function $f(x) = \sum_{n\geq 0} a_n x^n$. The choice algorithm implies:

$$f(x) = (x + x^2)(1 + f(x) + f(x)^2 + \cdots) = \frac{x + x^2}{1 - f(x)}.$$

Thus, $f(x)^2 - f(x) + (x{+}x^2) = 0$ and $f(x) = \frac{1}{2}(1{-}\sqrt{1{-}4x{-}4x^2})$.

2c. Wolfram give the x^20 series coefficient as: $a_{20} = 83,015,133,184$. Lagrange inversion is not directly applicable, since the recurrence equation cannot be put in the form $g(f(x)) = x$.

2d. Let $g(x)$ be the ordinary generating function of unlabeled ordered trees (without any consideration of single or couple nodes). Then our function $f(x)$ enumerating conjugal family trees is simply: $f(x) = g(x{+}x^2)$.

You should think of this as follows. In an ordinary labeled tree, the variable $x = x^1$ represents a node. To get a conjugal tree, replace each node by a single or couple node, substituting $x{+}x^2$ for x in the generating function. This kind of substitution is generally applicable when you replace the nodes of one family by the objects of another family.

3. Given that only couple nodes can have children, the recurrence becomes:

$$f(x) = x + \frac{x^2}{1 - f(x)}.$$

This shows how simple it is to adapt the recurrence mechanism to various rules for tree-construction. The rest is similar to Prob. 2.