Homework: math.msu.edu/∼magyar/Math482/Old.htm#2-24.

1a. We have $f(x)^2 - f(x) + x = 0$. By the Quadratic Formula:

$$f(x) = \sum_{n\geq 1} a_n x^n = \frac{1-\sqrt{1-4x}}{2}.$$

Comparing to the Catalan generating function $\sum_{n\geq 0} C_n x^n = \frac{1}{2x}(1-\sqrt{1-4x})$, we have $\sum_{n\geq 1} a_n x^n = x\sum_{k\geq 0} C_k x^k$, so $a_n = C_{n-1}$.

1b. Deletion Transform: deleting the ancestor of an ordered tree leaves a list of trees of the same type. Recursive choice algorithm:

$$(\text{Tree } T) \quad \Longleftrightarrow \quad (\text{root}) \;\text{ and }\; (\text{child trees: none or } (T_1) \text{ or } (T_1, T_2) \text{ or } \cdots)$$

Translating into algebra:

$$f(x) = x(1 + f(x) + f(x)^2 + f(x)^3 + \cdots) = x\,\frac{1}{1-f(x)}.$$

Clearing denominators gives $f(x) - f(x)^2 = x$, equivalent to Prob 1a.

Student Question: "I got $f(x) = 1/(1-f(x))$ from the choice algorithm. The solution says it should be $f(x) = x\cdot 1/(1-f(x))$, but I wasn't sure where the factor of x should come from. It says the factor of x comes from the choice of root in the algorithm, but if we're specifically getting rid of the ancestor, why is the root even a choice? "

Answer: When translating a choice algorithm into algebra, it is crucial to account for every case with its *correct weight* (i.e. with the correct power x^n).

Each n-vertex tree corresponds to the list of its child trees, but to give them the same weight n as the original tree, we must think of the child trees *together* with the disconnected root vertex.

That is, we only get an equality of generating functions if we put the extra factor $1x^1$ on the right side. The coefficient 1 means there is only 1 way to choose the disconnected root; the exponent x^1 means the root constitutes 1 extra vertex.

1c. For $g(x) = x - x^2$, we get: $g(f(x)) = f(x) - f(x)^2 = x$ by the previous. Thus, by definition, $g(x)$ and $f(x)$ are inverses. Note that this also works the other way:

$$f(g(x)) = \tfrac{1}{2}(1-\sqrt{1-4(x-x^2)}) = \tfrac{1}{2}(1-\sqrt{1-4x+4x^2}) = \tfrac{1}{2}(1-(1-2x)) = x.$$

2a. Deletion Transform: Deleting the ancestor of an ordered binary tree leaves either nothing or two trees of the same type. Recursive choice algorithm:
(Choose a tree) ⇔ (root) and either (nothing) or (T_1, T_2).

Generating function equation: $f(x) = x(1 + f(x)^2)$. Solving $xf(x)^2 - f(x) + x = 0$, we get $f(x) = \frac{1}{2x}(1-\sqrt{1-4x^2})$.

Comparing to the Catalan generating function, we find: $f(x) = \sum_{n\geq 1} a_n x^n = x\sum_{k\geq 0} C_k x^{2k}$, so $a_{2k+1} = C_k$, and $a_{2k} = 0$.

2b. From the equation of Prob 2a, we get $\frac{f(x)}{1+f(x)^2} = x$, which means $g(f(x)) = x$ for $g(x) = \frac{x}{1+x^2}$.

3. Paralleling Prob 2: $f(x) = x(1 + f(x)^3)$, so $f(x)$ is the solution of the equation $xf(x)^3 - f(x) + x = 0$. There is no neat solution to this equation, but $f(x)$ is the inverse function of $g(x) = \frac{x}{1+x^3}$.

4a. For every unrooted Cayley tree with $V = [n]$, we can circle any of the n vertices to designate a root, so the number of rooted Cayley trees is $(n^{n-2})(n) = n^{n-1}$.

4b. Deletion Transform: Deleting the root leaves a set of trees of the same type, with label sets to be tamped down. That is:

$$\text{(Choose a tree)} \quad \Longleftrightarrow \quad \text{(root) and child trees: (none or } \{T_1\} \text{ or } \{T_1, T_2\} \text{ or } \cdots).$$

Since the trees are labeled, we use the exponential generating function $f(x)$ and the Exponential Formula (Notes 2/7, Prop 4), getting the equation: $f(x) = xe^{f(x)}$.

4d. From Part (b), we see $f(x)/e^{f(x)} = x$, so $f(x)$ is the inverse function of $g(x) = x/e^x$.