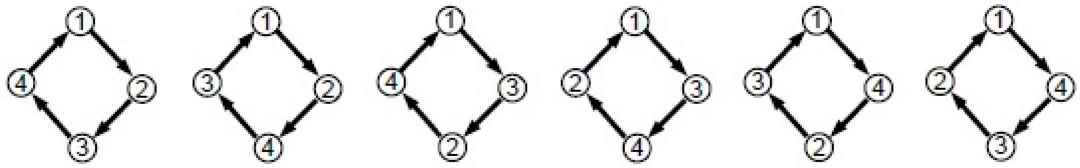
Student Name: Solutions

MTH 482 **Quiz 13** **Feb 7, 2014**

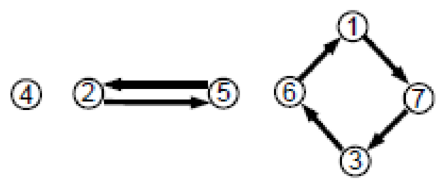
Consider the exponential family whose pictures are directed cycles of n labeled vertices (i.e., cycle-graphs with edges oriented clockwise). For example, the pictures of weight 4 are:



Also the pictures of weight 1 and 2 are: ① and ①⇆②. Since the label 1 can always be rotated to the top, and the remaining $n-1$ labels form a list after it, there are exactly $(n-1)!$ directed cycles for each weight $n \geq 1$.

1. What kind of objects are the hands in this family? Describe in words.

A card is a directed cycle with specified number labels replacing the standard labels. A hand is a set of cards whose label sets make $[n] = \{1, 2, \ldots, n\}$, with no repeated labels. For example, the hand:



is a set of 3 cards (directed cycles) with total label set $\{4\} \cup \{2, 5\} \cup \{1, 7, 3, 6\} = [7]$.

2. Give a simple formula for the deck-enumerator function $\tilde{d}(x)$ of this family, the exponential generating function of the deck-enumerator sequence $\{d_n\}_{n\geq 1}$.

The deck enumerator number d_n is the number of pictures of weight n, namely $d_n = (n-1)!$. The generating function is:

$$\tilde{d}(x) = \sum_{n\geq 1}(n-1)!\frac{x^n}{n!} = \sum_{n\geq 1}\frac{x^n}{n} = \log\left(\frac{1}{1-x}\right).$$

3. Give a simple formula for the hand-enumerator function $\tilde{h}(x)$ of this family, the exponential generating function of the hand-enumerator sequence $\{h_n\}_{n\geq 0}$.

By the Exponential Formula:

$$\tilde{h}(x) = \exp d(x) = \exp\log\left(\frac{1}{1-x}\right) = \frac{1}{1-x}.$$

This implies $h_n/n! = 1$, so $h_n = n!$, and indeed a hand can be thought of as the cycle notation for a permutation of $[n]$.