Steinitz Theorem: G a 3-connected planar graph $\Longrightarrow$ G edge-graph of convex polyhedron P.

Proof strategy:

1. Construct a "well-proportioned" planar drawing of G.
2. Lift it to 3 dimensions by giving a height to each vertex.

Today we concentrate on #1. To make precise the idea of "well-proportioned", we consider a physical situation in which we pin down an outer cycle of the graph, forming a convex polygon. We consider each internal edge as a rubber band exerting a Hooke force proportional to its length, pulling the end-vertices toward each other. The equilibrium position, in which all the edge-forces on internal vertices cancel out, is the well-proportioned drawing. This position is unique for a fixed arrangement of the external vertices.

Toy Model. We consider the case of 4 vertices on a line (not a plane), arranged in a path $v_0{-}v_1{-}v_2{-}v_3$. The end-vertices v_0, v_3 are external, pinned at $x_0 = 0$ and $x_3 = 1$; and the two internal vertices v_1, v_2 have positions x_1 and x_2. The forces on v_1 and v_2 are:

$$\begin{aligned} F_1(x_1,x_2) &= (0{-}x_1) + (x_2{-}x_1) \\ F_2(x_1,x_2) &= (x_1{-}x_2) + (1{-}x_2). \end{aligned}$$

For example, if $0 < x_1 < x_2 < 1$, then the forces on v_1 are $0{-}x_1$ (negative, toward v_0 at 0) and $x_2{-}x_1$ (positive, toward v_2 at x_2). The equilibrium position is the (x_1, x_2) for which both forces are zero. This is found by simplifying $F_1(x_1,x_2) = -2x_1 + x_2$ and $F_2(x_1,x_2) = x_1 - 2x_2 + 1$, and solving the linear system:

$$\begin{aligned} -2x_1 + x_2 &= 0 \\ x_1 - 2x_2 &= -1. \end{aligned}$$

Performing Gaussian Elimination gives $x_1{=}\frac{1}{3},\ x_2{=}\frac{2}{3}$, meaning that at equilibrium, the four points are equally spaced on the line, which is certainly a well-proportioned arrangement.

Potential Function. We can find the equilibrium another way, which gives more qualitative insight about why it is unique. The potential energy is the work done against the force to move two particles from $(0,0)$ to two particular values (x_1,x_2): that is, $PE(x_1,x_2) = -\oint_{(0,0)}^{(x_1,x_2)} F(\vec{r}) \cdot d\vec{r}$. It turns out that:

$$PE(x_1,x_2) \;=\; \tfrac{1}{2}\sum_{ij\in E}(x_i - x_j)^2 = \tfrac{1}{2}\left((0{-}x_1)^2 + (x_1{-}x_2)^2 + (x_2{-}1)^2\right),$$

where E is the edge-set of our path graph. It is clear that $\nabla PE = (\frac{\partial}{\partial x_1}PE, \frac{\partial}{\partial x_2}PE) = -(F_1, F_2)$, so it must be correct (except possibly for the constant term).[1]

[1] We can compute this by an integral along the straight-line path in $\mathbb{R}^4$ which moves all four vertices from the origin to their assigned positions: $\vec{r}(t) = (tx_0, tx_1, tx_2, tx_3) = (0, tx_1, tx_2, t)$ for $0 \le t \le 1$:

$$\begin{aligned} PE(x_1,x_2) &= -\int_{t=0}^{1}(F_0,F_1,F_2,F_3)\cdot \vec{r}\,'(t)\,dt \\ &= -\int_{t=0}^{1}(F_0,F_1,F_2,F_3)\cdot (x_0,x_1,x_2,x_3)\,dt \\ &= \tfrac{1}{2}\big(\,(x_0{-}x_1)x_0 + (x_1{-}x_0)x_1 + (x_1{-}x_2)x_1 + (x_2{-}x_1)x_2 + (x_2{-}x_3)x_2 + (x_3{-}x_2)x_3\,\big). \end{aligned}$$

Each edge v_iv_j contributes twice to this sum, adding a total of:

$$(x_i{-}x_j)x_i + (x_j{-}x_i)x_j \;=\; (x_i{-}x_j)x_i - (x_i{-}x_j)x_j \;=\; (x_i{-}x_j)^2,$$

giving the stated formula for PE.

Now, it known that any such quadratic function can be diagonalized into the form:

$$g(x_1, x_2) = a_1(b_1x_1 + c_1x_2 - d_1)^2 + a_2(b_2x_1 + c_2x_2 - d_2)^2 + d.$$

The shape of such a function is either:

(+) an upward-curving paraboloid, if $a_1, a_2 > 0$;

(0) a saddle-surface, if a_1 and a_2 have opposite signs

(−) a downward-curving paraboloid, if $a_1, a_2 < 0$.

That is, g is either positive definite, indefinite, or negative definite.

Which is the category of our function PE? It is clear from the sum-of-squares form of $PE(x_1, x_2)$ that if $|x_1|$ or $|x_2|$ is large, then $PE(x_1, x_2)$ is a large positive value.

Conclusion. PE must be positive definite (+), which means it has a *unique* critical point where $\nabla PE(x_1, x_2) = (0, 0)$, namely its minimum point; and this is the unique equilibrium point where $(F_1, F_2) = (0, 0)$.