The Number of Three-Dimensional Convex Polyhedra

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Abstract

A convex polyhedron, or polytope, is the bounded intersection of closed half-spaces. The problems of determining the number of three dimensional convex polyhedra as a function of the number of faces or edges or both have been around for over 150 years. Except for Steinitz's conversion of polyhedra to "planar maps", little was done on the problem until the work on "rooted" planar maps in the 1960's. Recently the original (unrooted) questions have been answered asymptotically. We will retrace the steps that led to this result.

1. Introduction

Convex polyhedra (also called polytopes) are the analogues to convex polygons in higher dimensions. We can define a convex polyhedron as a bounded intersection of closed half-spaces. Alternatively, we could define a convex polyhedron to be the convex hull of a finite set of points. We will be concerned exclusively with three-dimensional polyhedra: those that lie in 3-space but do not lie in a plane. In the future, "polyhedra" will always mean "three-dimensional convex polyhedra." Cubes, tetrahedra, and prisms are all examples of polyhedra. In an obvious way, polyhedra have vertices, edges, and faces. (These can be described formally, but we need not do so here.) We will use the notation P_0, P_1, and P_2 for the numbers of vertices, edges, and faces, respectively, of a polyhedron P. Euler's famous theorem (1752) states that

$$P_0 - P_1 + P_2 = 2.$$

Suppose that v, e, and f are positive integers with $v - e + f = 2$. Does there exist a polyhedron P with $P_0 = v$, $P_1 = e$, and $P_2 = f$? The answer is No in general; however, it is easy to give necessary and sufficient conditions:

THEOREM 1. *There is a convex polyhedron P with*

$$P_0 = v, \qquad P_1 = e, \quad \text{and} \quad P_2 = f$$

if and only if

$$v - e + f = 2, \quad \text{and} \quad 4 \leqslant v \leqslant 2e/3, \quad \text{and } 4 \leqslant f \leqslant 2e/3.$$

Edward A. Bender: Working in the area where matrix theory and number theory meet, I received my Ph.D. under Olga Taussky at Caltech in 1966. I was a Peirce Instructor at Harvard and a member of the research staff at the Communications Research Division of the Institute for Defense Analyses before I joined the University of California in 1974. My major professional interest at present is asymptotic enumeration. I flirt with other areas of "concrete" mathematics, population biology, and computer science.

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Proof. The first condition is Euler's Theorem. Since the "smallest" polyhedron is a tetrahedron, the lower bounds on v and f are necessary. An edge corresponds to an unordered pair of vertices, called its ends. Every vertex must be the end of at least three edges, and each edge has two ends. If t is the total number of ends, then $3v \leqslant t = 2e$ and so $v \leqslant 2e/3$. Similarly, since each face is bordered by at least three edges and each edge lies on two faces, $f \leqslant 2e/3$.

Here's a sketch of the converse. If the triple (v, e, f) satisfies the conditions, then there are nonnegative integers x and y such that

$$(v, e, f) - x(1,3,2) - y(2,3,1)$$

equals one of $(6,10,6)$, $(6,9,5)$, $(5,9,6)$, and $(4,6,4)$. (Prove it by induction on e.) Each of the four triples just listed can be realized by a polyhedron, which we'll call irreducible. Adding $(2,3,1)$ can be realized by slightly adjusting the edges meeting at a vertex and replacing the vertex by a triangular face. Adding $(1,3,2)$ can be realized similarly by introducing two triangular faces in place of two adjacent edges of some face. (See Figure 1.) By iterating, one of the four irreducible polyhedra can be built up to realize (v, e, f). □

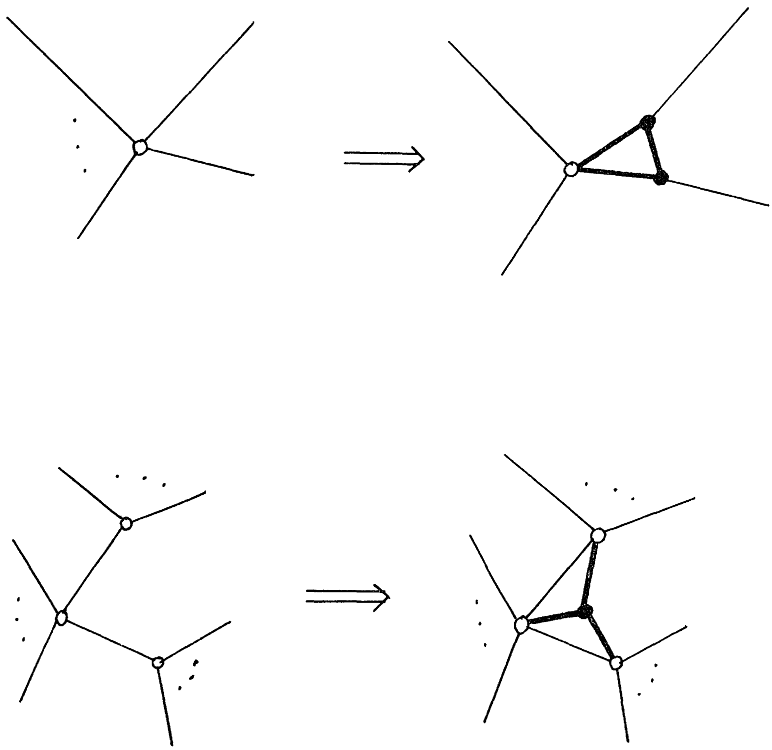

FIG. 1. Building up to a specified (v, e, f). Additions are shown heavy.

Since the existence question was easily settled, we proceed to the next type of question:

Q. How many "distinct" convex polyhedra P have $P_0 = v$ and $P_2 = f$?

There is no need to specify P_1 since it equals $v + f - 2$. By restricting only one of the P_k's, we are led to ask for $k = 0, 1$ and 2.

Q_k. How many "distinct" convex polyhedra P have $P_k = n$?

To answer the questions, we need to say what we mean by distinct polyhedra. Given two polyhedra P and Q, there may be a one-to-one mapping m of the faces of P to the faces of Q that preserves incidence; i.e., if F_1 and F_2 are faces of P that intersect in exactly one edge (resp., vertex), then $m(F_1)$ and $m(F_2)$ intersect in exactly one edge (resp., vertex), and conversely. If no such mapping exists, P and Q are called *combinatorially distinct*. This will be what we mean by "distinct."

Steiner posed Q_2 in 1832 and Kirkman stated in 1878 that he saw no hope of answering Q with the present power of mathematics. Shepard (1968) asked for a close approximation to the answer to Q_0.

The *dual*, P^*, of a polyhedron P is constructed by placing a vertex in each face of P^* and joining two such vertices by an edge if and only if the corresponding faces of P share an edge. The faces of P^* correspond to the vertices of P. One can show that $P^{**} = P$. Thus duality is a bijection, Q_0 and Q_2 have the same answer, and the answer to Q remains unchanged if the values of v and f are switched.

For specific values of v and f (resp., k and n), the corresponding Q (resp., Q_k) can be answered in a finite length of time since an algorithm exists for constructing all polyhedra with given parameters. This method is not an acceptable answer. What is an *acceptable answer*? One possible definition, suggested by the theory of algorithms, is: a way of calculating the number, which requires an amount of time that is a polynomial in v and f (or n). It is quite likely that no answer in this sense exists. How can we relax the definition of an answer? One way is to allow more time for calculating the formula. If this is relaxed too much, one can use the algorithm alluded to earlier for generating all polyhedra.

Another way to adjust the notion of answer is by requiring only a good approximate formula that can be computed quickly. What is a *good approximation*? We will require that the percentage error in the approximation go to zero as v and f (or n) get large. Such answers, which give information about how numbers behave as the parameters get large, are called *asymptotic* formulas. All the questions Q, Q_0, Q_1, Q_2 above have now been answered asymptotically.

In this paper we will retrace the path to the answers. The first step was taken by Steinitz (1922), who converted the questions to problems about counting graphs in the plane. Nothing further was done until Tutte developed methods for planar enumeration in the 1960s. As a result, Mullin and Schellenberg (1968) obtained a "generating function" for "rooted" polyhedra with given numbers of vertices and faces. This led to an explicit but messy formula for rooted polyhedra. Bender and Richmond (1984) used the generating function to obtain an asymptotic formula that is valid for part of the range of v and f. Bender and Wormald (1985) combined this with various estimates to show that an asymptotic answer to Q or Q_k for rooted polyhedra gives an answer for polyhedra. The entire range was covered by Bender and Wormald (to appear) as a result of work on the paper you are now reading.

If all proofs had been included, this article would have been a small monograph. Therefore, I have replaced most proofs with broad sketches. If you are interested in the details, consult the original articles.

Federico (1975) was the source of the historical information. For a variety of questions concerning polyhedra, see Shepard (1968).

2. The graph-theory problem

A *graph* consists of a set of vertices with edges joining some pairs of the vertices. The vertices and edges are not labeled. If there is a path from every vertex to every other vertex along the edges, then the graph is called *connected*. A *loop* is an edge with both endpoints the same. If the ends of e_1 are the same as the ends of e_2, we say that e_1 and e_2 constitute a *multiple edge*. A (*planar*) *map* is a connected graph drawn on the plane so that no edges cross. The maximal regions containing no edges are called *faces*. The unbounded face is called the *external face*. Two maps are considered the same if one can be converted into the other by stretching, contracting and/or reflecting the plane.

A map is called *k-connected* if there *does not exist* a positive integer $j < k$ and a partition of the edges into two sets E_1 and E_2 such that each set contains at least j edges and the edges in $E_1 \cap E_2$ contain at most j distinct endpoints. Here are some simple useful observations on k-connectedness.

O1. "Connected" is the same as "1-connected."

O2. A map with at least 2 edges is 2-connected if and only if it contains no loops and no vertex is encountered more than once as we walk around a face boundary.

O3. A 2-connected map with at least 4 edges is 3-connected if and only if it contains no multiple edges and every pair of faces that have two vertices in common, say v and w, also have the edge (v, w) in common.

Exercise 1. Prove the observations. To prove O2, note that if v is encountered twice then its removal disconnects the graph. To prove O3, note that removal of v and w splits the boundary of each face containing both v and w. If (v, w) is not a common edge of these faces, this disconnects the graph. You may find it easier to see what is happening if you draw the map so that one of those faces is external.

A polyhedron may be converted to a map as follows. Select a face F. Remove all of the polyhedron except the edges and vertices. Place a plane parallel to F on the opposite side of the polyhedron from F. Place a light outside the polyhedron near the center of F. If the light is placed carefully, none of the shadows that the edges cast on the plane will cross each other and the shadow of the boundary of F will bound the external face of a map formed by the shadows. The faces of the polyhedron correspond to the faces of the map. See Figure 2. This picture is called a *Schlegel diagram* for the polyhedron.

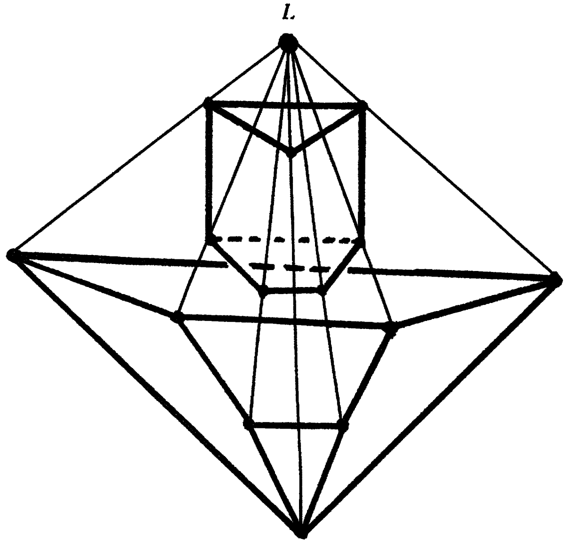

FIG. 2. A polyhedron projected down onto a plane by the light L gives a Schlegel diagram.

THEOREM 2. (Steinitz) *A map is the Schlegel diagram of a convex polyhedron if and only if it has at least* 4 *vertices and is* 3-*connected*.

Exercise 2. Prove the necessity by using O3.

The converse is difficult. For a proof, see Chapter 13 of Grünbaum. The lack of a corresponding result in higher dimensions seriously hampers attempts to count convex polyhedra in those dimensions. □

Unfortunately, there are generally many Schlegel diagrams for each polyhedron. This leads us to the notion of *rooted* maps and polyhedra. A polyhedron is rooted by choosing an edge (called the *root edge*), one vertex on the edge (called the *root vertex*) and one face adjacent to the root edge (called the *root face*). A 2-connected map is rooted if an edge on the external face (also called the root face) is distinguished. The *root-face degree* of a map or polyhedron is the number of edges on the root face.

COROLLARY 2.1. *There is a one-to-one correspondence between rooted convex polyhedra and rooted 3-connected maps with at least 4-vertices.*

Proof. The direction of a root edge will be such that the root vertex is the tail of the root edge. Arrange the polyhedron so that when the root face is viewed from the outside and traversed in the direction of the root edge, the traversal is clockwise. (This may require reflecting the polyhedron.) Now place the light near the center of the root face. This gives a one-to-one correspondence. □

In the next section we will discuss the generating function for 3-connected maps. In Sections 4 and 5 we will see how that leads to asymptotics for rooted Schlegel diagrams and, hence, for rooted polyhedra. If all the ways of rooting a polyhedron P were distinct, it would have $P_1 \cdot 2 \cdot 2 = 4P_1$ rooted versions. In Section 6 we will see that for most polyhedra all rootings are distinct. This provides us with asymptotic answers to the questions Q and Q_k.

3. The exact number of rooted maps

Generating functions. If a_n is a sequence defined for $n \geqslant 0$, then the *generating function* for the sequence is

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We will adopt the convention of using a lower-case letter for a sequence element and the corresponding upper-case letter for the corresponding generating function. These ideas extend to multiply indexed sequences; for example, the generating function for the triply indexed sequence $b_{i,j,k}$ is

$$B(x, y, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{i,j,k} x^i y^j z^k.$$

All the infinite series that we use converge when the variables are sufficiently small.

Suppose that we are given a generating function for some sequence, say, $a_{i,j}$. By Taylor's Theorem for functions of two variables, the coefficients of the power series for $A(x, y)$ are uniquely determined and so must be the sequence $a_{i,j}$. Thus, if we somehow explicitly expand $A(x, y)$ in a power series, we will obtain a formula for the sequence $a_{i,j}$.

There are a variety of rooted maps that one might enumerate. Each of these problems is approached by describing a method for constructing maps out of other maps. When this description is translated into a statement involving generating functions, the result is a functional relationship among the generating functions. If the construction involves only the type of map we are counting, then the functional equation involves only the generating function we are interested in. Thus it can be solved, at least in principle, for that generating function.

2-*connected maps*

Brown and Tutte (1964) enumerated 2-connected rooted maps. Since their result is central in later calculations, we'll look at their method.

Suppose M is a rooted 2-connected map that is not a single edge. The root edge of M belongs to two faces, the exterior face and some interior face, which are

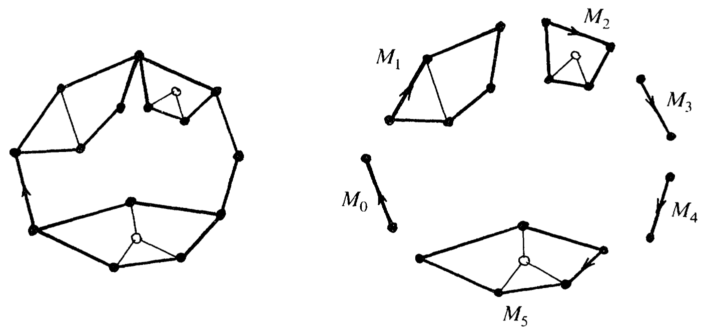

FIG. 3. Decomposing a 2-connected map.

shown by heavy lines in Figure 3. By splitting the map into pieces at all vertices v such that v lies on both faces, we obtain a sequence of maps $M_0, \ldots, M_t$, where M_0 is the root edge. By O2, each M_t can be seen to be 2-connected. Each can be rooted by rooting the edge of M_i first encountered when following the external face in a clockwise direction starting from the root edge. This decomposition is reversible provided we specify which edge on the root face of each M_i is the last edge encountered on the root face of M in our clockwise traversal. Therefore, we have a unique method for building up a 2-connected map from 2-connected maps with fewer faces.

In order to describe this numerically, we need to keep track of the number of vertices, the number of faces, and the degree of the root face. The last quantity is needed because an M_t with root-face degree k has $k - 1$ possible choices for the last vertex on the root face of M.

Let $f_{i,j,k}$ be the number of rooted 2-connected maps with $i + 1$ vertices, $j + 1$ faces, and root-degree k, except that the map consisting of a single edge is not counted. Since the root-face degree does not interest us, we want $F(x, y, 1)$, the generating function for $f_{i,j} = \sum_k f_{i,j,k}$. It can be shown that the above construction is equivalent to the equation

$$F(x, y, z) = yz \sum_{t=1}^{\infty} \left(\sum_{i,j,k} f_{i,j,k} x^i y^j (z + z^2 + \cdots + z^{k-1}) + xz \right)^t \tag{3.1}$$

After a little manipulation we get

$$F(x, y, z) = yz \sum_{t=0}^{\infty} \left((zF(x, y, 1) - F(x, y, z))/(1 - z) + xz \right)^t - yz.$$

After summing the geometric series, rearranging, and writing F for $F(x, y, z)$ and F_1 for $F(x, y, 1)$, we obtain

$$F^2 + ((1 - z)(1 - xz) + yz - zF_1)F - yz^2(x - xz + F_1) = 0. \tag{3.2}$$

Since (3.1) is a direct translation of a construction which builds up maps out of maps with fewer faces, (3.2) and the initial conditions $f_{i,0,k} = 0$ must determine $F(x, y, z)$ uniquely. Since both F and F_1 appear in (3.2), it is not clear how to extract F or F_1 from (3.2) (setting $z = 1$ simply leads to the equation $0 = 0$). One approach is to use educated guessing, as done by Brown and Tutte. There is a more systematic approach (Brown, 1968), but it can lead to a morass of algebra: Complete the square in (3.2) to obtain an equation of the form

$$(F + \text{stuff})^2 = G(x, y, z, F_1). \tag{3.3}$$

Let $z = Z(x, y)$ stand for the value of z for which the left side of (3.3) vanishes. Since the left side of (3.3) is a square, its derivative with respect to z also vanishes at Z. Applying this to the right side of (3.3) we obtain the two equations

$$G(x, y, Z, F_1) = 0 \quad \text{and} \quad G_z(x, y, Z, F_1) = 0$$

in the two unknowns Z and F_1. These are rational equations, and they can be "solved" for F_1.

THEOREM 3. (Brown and Tutte) *The number $f_{i,j}$ of rooted 2-connected maps with $i + 1$ vertices and $j + 1$ faces is given by*

$$F(x, y) = uv(1 - u - v),$$

where u and v are given implicitly by

$$x = u(1 - v)^2, \qquad y = v(1 - u)^2, \quad \text{and} \quad u(0,0) = v(0,0) = 0.$$

The explicit values are

$$\frac{(2i + j - 2)!(2j + i - 2)!}{i!j!(2i - 1)!(2j - 1)!}.$$

Proof. Since $F(x, y) = F(x, y, 1)$, we could solve $G = G_z = 0$ for F_1. This is done by parameterizing x and y as stated in the theorem. The explicit value for $f_{i,j}$ is obtained by a technique known as *Lagrange inversion*. That technique expresses the coefficients of any function $H(u(x, y), v(x, y))$ in terms of derivatives of H and of $u(1 - v)^2$ and $v(1 - u)^2$ with respect to u and v. See Section 5 for a discussion. □

Quadrangulations

A *quadrangulation* is a map such that each internal face is a quadrilateral.

There is a connection between rooted 2-connected maps and rooted quadrangulations with quadrilateral external faces, discovered by Brown (1965). Let M be a rooted map with vertex set V and face set F. Place a vertex in the middle of each face to form a new set of vertices V^*. Define a set of edges by connecting v and v^* if and only if v is a vertex of the face corresponding to v^*. This gives a map Q. Let r be the root vertex of M, $e = (r, v)$ the root edge of M and v^* the vertex corresponding to the external face of M. The root vertex of Q is v and the root edge

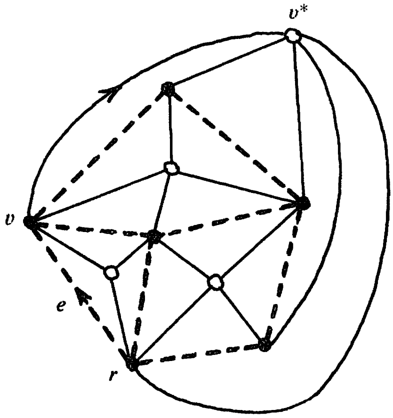

FIG. 4. Heavy dashed lines are the edges of the original 2-connected map. Solid lines are the edges of the corresponding quadrangulation.

is (v, v^*). The edges joining v^* are drawn so that no edges of Q lie outside of the region bounded by (v, v^*), (v^*, r) and e. See Figure 4.

THEOREM 4. *The above correspondence is a bijection between rooted* 2*-connected graphs with more than one edge and rooted quadrangulations with quadrilateral external faces. Furthermore, the original graph is* 3*-connected if and only if each* 4*-cycle of edges in the quadrangulation is a face.*

Proof. The bijection is due to Brown (1965, Sec. 7) and the last part of the theorem is due to Mullin and Schellenberg (1968, Sec. 5).

Exercise 3. Construct proofs using O2 and O3. □

Call a quadrangulation that corresponds to a 3-connected map *simple*.

Exercise 4. Show that the vertices of a quadrangulation can be partitioned uniquely into two sets, called red and green, such that edges connect only vertices of different colors, the red vertices correspond to the vertices of the corresponding 2-connected map and the green vertices to the map's faces.

Note that the number of quadrangulations with root degree 4, $i + 1$ red vertices and $j + 1$ green vertices equals $f_{i,j}$, the number of 2-connected rooted maps with $i + 1$ vertices and $j + 1$ faces. Let $p_{i,j}$ be the number of those that are simple and have at least 8 vertices. By Theorem 4 and Corollary 2.1, the function $P(x, y)$ counts rooted polyhedra. (The condition on vertices in $p_{i,j}$ eliminates small 3-connected maps that do not correspond to polyhedra.)

A quadrangulation has a *diagonal* if there are external vertices v and w and an internal vertex x so that (v, x) and (w, x) are edges. Let $n_{i,j}$ count the number of quadrangulations counted by $f_{i,j}$ that have no diagonals and let $N(x, y)$ be the corresponding generating function.

Every quadrangulation with root degree 4, more than 4 vertices and no diagonals can be built from a simple quadrangulation having at least 6 vertices. This is done by replacing the internal faces of the simple quadrangulation with arbitrary quadrangulations of root degree 4. Mullin and Schellenberg (1968, Sec. 6) show that this construction is uniquely determined and corresponds to the generating function equation

$$(xy/F)P(F/y, F/x) = N(x, y) - xy. \tag{3.4}$$

The quadrangulations counted by $f_{i,j}$ can be broken into three disjoint classes:

(i) no diagonals, counted by $N(x, y)$;
(ii) diagonal at the root, counted by, say, $R(x, y)$;
(iii) diagonal not at the root, counted by, say, $D(x, y)$.

Thus,

$$F(x, y) = N(x, y) + R(x, y) + D(x, y).$$

By the type of construction shown in Figure 5 it follows that

$$R(x, y) = (N(x, y) + D(x, y))F(x, y)/x.$$

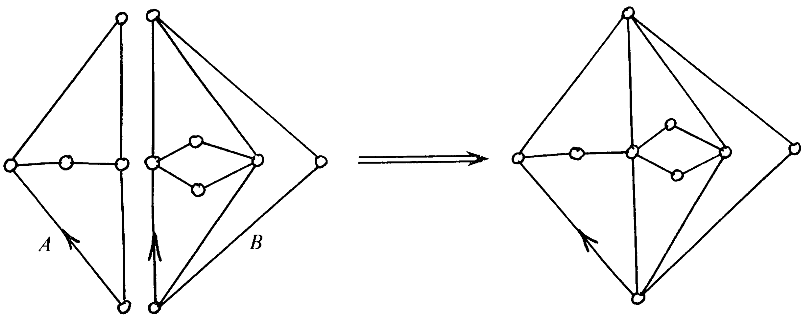

FIG. 5. Building up $R(x, y)$. Quadrangulation A has no root diagonal but B may have any number.

Interchanging the roles of red and green vertices (see Exercise 4) gives

$$D(x, y) = (N(x, y) + R(x, y))F(x, y)/y.$$

By performing some algebraic manipulations on the last three equations, one can show that

$$N = \left((1 + F/y)^{-1} + (1 + F/x)^{-1} - 1\right)F. \tag{3.5}$$

By setting $X = F/y$ and $Y = F/x$ and combining (3.4) and (3.5):

$$P(X, Y) = \left((1 + X)^{-1} + (1 + Y)^{-1} - 1\right)XY - F(x, y).$$

Exercise 5. Using this with Theorem 3 and setting $r = u/(1 - u - v)$ and $s = v/(1 - u - v)$, show

THEOREM 5. (Mullin and Schellenberg) *The generating function for the number of distinct rooted convex polyhedra with* $i + 1$ *vertices and* $j + 1$ *faces is given by*

$$P(X, Y) = \left((1 + X)^{-1} + (1 + Y)^{-1} - 1\right)XY - F,$$

where

$$F = rs/(r + s + 1)^3$$

and r *and* s *are given implicitly by*

$$r = X(s + 1)^2, \qquad s = Y(r + 1)^2, \qquad r(0,0) = s(0,0) = 0.$$

Mullin and Schellenberg applied Lagrange inversion to obtain a formula for $p_{i,j}$. Unfortunately, it is a double summation with alternating signs, so it seems hard to see how $p_{i,j}$ behaves except by computing specific values. I'll say more about Lagrange inversion and an exact formula in Section 5.

Note that the generating function for $P(X, Y)$ is symmetric in X and Y. This result also follows immediately from duality without ever seeing the generating functions. Various other generating functions follow easily from $P(X, Y)$. Here are two examples. The coefficient of Y^k in $P(1, Y)$ is the number of rooted convex polyhedra with k faces. By Euler's theorem, the coefficient of $x^i y^n$ in $P(xy, y)$ is the number of convex polyhedra with $i + 1$ vertices and n edges.

4. The asymptotic number of rooted polyhedra

We need some notation for writing asymptotics.

$$f(n) \sim g(n) \text{ means that } f(n)/g(n) \to 1 \text{ as } n \to \infty;$$

$$f(n) = O(g(n)) \text{ means that } f(n)/g(n) \to 0 \text{ as } n \to \infty.$$

By convention, $f(n)/g(n) = 1$ when $f(n) = g(n) = 0$. For functions of two (or more) variables, the terminology is more involved. Let R be a region in the xy-plane containing integer points (x, y) with $\min(x, y)$ arbitrarily large. We say that

$$f(m, n) \sim g(m, n) \text{ uniformly in } R$$

if

$$\sup|f(m, n)/g(m, n) - 1| \to 0 \text{ as } k \to \infty,$$

where the supremum is taken over all (m, n) in R with $\min(m, n) \geqslant k$. We define $f(m, n) = o(g(m, n))$ uniformly in R in a similar fashion.

There are several possible approaches to obtaining asymptotics for rooted maps. The most straightforward is to work with a simple formula like that for $f_{i,j}$ in Theorem 3 together with Stirling's formula,

$$n! \sim (2\pi n)^{1/2}(n/e)^n. \tag{4.1}$$

This idea can also be adapted to certain types of sums, but seldom to those with alternating signs unless the initial terms dominate the sum. In cases like $P(X, Y)$, one tries to work directly with the generating function. For an introduction to asymptotics in combinatorics, see Bender (1974).

We now turn to $P(X, Y)$. Think of it as a function of two complex variables X and Y. Such functions have places at which they misbehave, called "singularities." A little bit of knowledge about the nature of the singularities closest to the origin is often sufficient to provide information about the coefficients of the power series for the function. It would take too much space to define singularities and discuss the connection between their nature and the coefficients of the power series. In this way Bender and Richmond (1984) obtained messy asymptotic formulas from $P(X, Y)$, $P(1, Y)$, $P(xy, y)$, etc. The result for $p_{i,j}$ was valid for $i \to \infty$ provided $1/2 + c < i/j < 2 - c$. This result leaves a gap at each end because i/j is constrained to stay away from $1/2$ and 2 while v/f could approach either $1/2$ or 2 as f gets large. The extreme ends were filled in by

THEOREM 6. (Tutte, 1962) *Let t_i be the number of rooted convex polyhedra with $i + 1$ vertices and all faces triangular. Then*

$$t_i = \frac{2(4i-7)!}{(3i-4)!(i-1)!} \sim \frac{3}{16(6\pi i^5)^{1/2}}\left(\frac{256}{27}\right)^{i-1}.$$

The same formula holds if t_i counts rooted convex polyhedra with $i + 1$ faces and all vertices of degree 3.

5. Reconsideration

While writing this paper, I simplified the messy asymptotic formula for $p_{i,j}$ mentioned earlier. Using this result and Theorem 6 as a guide, I conjectured

THEOREM 7. (Bender and Wormald, to appear) *Uniformly as $\min(i, j) \to \infty$*

$$p_{i,j} \sim \frac{1}{3^5 ij}\binom{2i}{j+3}\binom{2j}{i+3}.$$

Proof. No high powered tools are needed. By applying Lagrange inversion (see below) to Theorem 5 in a slightly different way than Mullin and Schellenberg did, a singly indexed summation is obtained. To make asymptotic calculations easier, the sum is transformed twice by using the Pascal triangle identity

$$\binom{c+1}{k} = \binom{c}{k} + \binom{c}{k-1}$$

and rearranging terms.

What is Lagrange inversion? Suppose we want the coefficient of x^n in $f(g(x))$ where the power series for $f(y)$ is known but that for $g(x)$ is not known. Instead, we only know the power series for the inverse function $g^{-1}(z)$. Lagrange inversion tells us how to compute the answer. There are various generalizations to functions of several variables. See (S. A. Joni, 1977) for a discussion and also a proof of the following.

THEOREM 8. (I. J. Good) *Suppose that* $f(z_1,\dots,z_k)$ *and* $H_i(z_1,\dots,z_k)$, $1 \leqslant i \leqslant k$ *are power series such that the* H_i*'s have nonzero constant terms. Let* $h_i = z_i H_i$. *Then there exist unique power series* $g_j(x_1,\dots,x_k)$ *satisfying the set of equations* $h_i(g_1,\dots,g_k) = x_i$. *Also, the coefficient of* $x_1^{n_1}\cdots x_k^{n_k}$ *in* $f(g_1,\dots,g_k)$ *equals the coefficient of* $z_1^{n_1}\cdots z_k^{n_k}$ *in*

$$\det(\partial h_i/\partial z_j) f/(H_1^{n_1+1}\cdots H_k^{n_k+1}).$$

This can be applied to Theorem 5 with

$$(z_1,z_2)=(r,s),\qquad (x_1,x_2)=(X,Y),\qquad (n_1,n_2)=(i,j),$$
$$f=-rs/(r+s+1)^3,\qquad h_1=r/(s+1)^2,\qquad h_2=s/(r+1)^2.$$

Exercise 6. Show that for $i, j > 1$, $p_{i,j}$ is the coefficient of $r^i s^j$ in

$$(r+s+1)^{-3}(s+1)^{2i-3}(r+1)^{2j-3}(3rs-r-s-1).$$

If we write

$$(r+s+1)^{-k}=\sum_{t,u}\binom{-k}{t}\binom{t}{u}r^u s^{t-u},$$

then Mullin and Schellenberg's formula is obtained. If we write

$$\begin{aligned}(r+s+1)^{-k}&=(r+s)^{-k}(1+r/(1+s))^{-k}\\&=\sum_j\binom{-k}{j}r^j(1+s)^{-k-j},\end{aligned}\tag{5.1}$$

then Bender and Wormald's formula is obtained. You may wish to carry out these calculations. If so, write

$$(1+r+s)^{-3}(3rs-r-s-1)=3rs(r+s+1)^{-3}-(r+s+1)^{-2}$$

and use (5.1) with $k=2$ and $k=3$.

6. The asymptotic number of polyhedra

In this section we will discuss the proof and application of the following theorem.

THEOREM 9. (Bender and Wormald, 1985) *Let* $u_{i,j}$ *be the number of unrooted convex polyhedra with* $i+1$ *vertices and* $j+1$ *faces. There are constants* A *and* $0<c<1$ *such that for all i and j,*

$$1 \leqslant 4(i+j)u_{i,j}/p_{i,j} < 1+c^i,$$

where $0/0$ *is interpreted as* 1.

The theorem says that $u_{i,j}$ approaches $p_{i,j}/4(i+j)$ very quickly. Thus Theorems 7 and 9 answer question Q. Answering Q_k involves estimating

$$\frac{1}{4\cdot 3^5}\sum\frac{1}{ij(i+j)}\binom{2i}{j+3}\binom{2j}{i+3},$$

where the sum ranges over appropriate values of i and j. This can be done by standard methods as discussed in Bender (1974) or by using results in Bender and Richmond (1984). The answers are

$$Q\sim\frac{1}{972ij(i+j)}\binom{2i}{j+3}\binom{2j}{i+3}=A(i,j),\text{ say;}$$

$$Q_0\sim\left(\pi n(4+\sqrt{7})/4\sqrt{7}\right)^{1/2}A\left(n-1,(n-1)(3+\sqrt{7})/4\right);$$

$$Q_1\sim\frac{\sqrt{\pi n}}{4}A(n/2,n/2);$$

$$Q_2=Q_0,$$

where fractional factorials in binomial coefficients are approximated by Stirling's formula (4.1). Tutte (1963) conjectured and Richmond and Wormald (1982) proved the asymptotic formula for Q_1.

How is Theorem 9 proved? As noted at the end of Section 2, there are $4(i+j)$ distinct ways to root a convex polyhedron with $i+j$ edges and no symmetries. If there are symmetries, the number of rootings is less. That accounts for the left-hand inequality in Theorem 9.

The right-hand inequality is based on estimates for rooted polyhedra with symmetries. There are three different types of symmetries possible for a polyhedron. One type preserves the orientation of the polyhedron and is essentially a rotation. The other two types reverse the orientation and are distinguished by whether or not the symmetry maps any vertex or edge into itself. If it has such an invariant, it can essentially be viewed as a reflection in a plane; otherwise, a reflection in a point.

For each type of symmetry, if you are given (i) a connected piece cut out of the polyhedron whose images under the symmetry cover the entire polyhedron and (ii) the nature of the symmetry, then you can reconstruct the entire polyhedron. The piece cut out for you may involve edges and faces that have been cut in half. The cut may also run through some vertices. Connect all those vertices to a single new vertex v and extend the cut edges to v. If the original cut is chosen carefully, then the resulting figure will be 3-connected. Since the number of vertices and edges in the new 3-connected graph is less than that in the original polyhedron, Theorem 7 can usually be used to obtain a crude but adequate upper bound. Various adaptations of this idea are needed to handle all the cases that arise. If you wish details, see the original paper.

Special cases of Theorem 9 were proved by Tutte (1980) and Richmond and Wormald (1982).

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Combinatorial and Functional Identities in One-Parameter Matrices

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1. Introduction. A one-parameter matrix is one whose entries depend on a parameter in such a way that matrix multiplication corresponds to performing an operation, e.g., addition or multiplication, on the parameter. More formally, if a

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