

Proposition (Hall Matching) *Let $G = (A \cup B, E)$ be a bipartite graph satisfying Hall's Condition: for any subset $A' \subset A$, we have $|A'| \leq |N(A')|$. Then there exists a matching from A to B .*

Here $N(A') \subset B$ denotes the neighbor set of A' . A *matching* from A to B means a set of edges $M \subset E$ such that every $a \in A$ lies on exactly one edge of M , and every $b \in B$ lies on at most one edge of M .

Proof: We first convert G into a flow network $\hat{G}$ whose directed graph includes G with all edges oriented from A to B ; an extra source vertex s with edges (s, a) for all $a \in A$; and an extra sink vertex t with edges (b, t) for all $b \in B$. We set the capacity of each edge to $c = 1$.

A partial matching M' from $A' \subset A$ to B clearly corresponds to a flow f in $\hat{G}$: to each edge $(a, b) \in M'$ there corresponds $f(s, a) = f(a, b) = f(b, t) = 1$, with all other $f(x, y) = 0$. Furthermore, $|A'| = \text{Flow}(f)$. Thus a matching of a maximal subset $A' \subset A$ corresponds to a maximal flow in $\hat{G}$.

Now, by the Max-Flow/Min-Cut Theorem, this is the same as the minimum capacity of a cut (S, T) :

$$\min_{(S, T)} \text{Cap}(S, T) = \max_{A'} |A'|.$$

(Recall that a pair of disjoint vertex sets (S, T) is a cut if $s \in S$, $t \in T$ and $S \cup T = A \cup B \cup \{s, t\}$. Also, $\text{Cap}(S, T) := |E(S, T)|$, the number of directed edges in $\hat{G}$ from S to T .) Thus, a complete matching ($A' = A$) is possible for G exactly when the minimal capacity for a cut of $\hat{G}$ is $n := |A|$.

Let (S, T) be any cut, and denote $S_A := S \cap A$, etc., so that $A = S_A \cup T_A$ and $B = S_B \cup T_B$. The capacity is:

$$\text{Cap}(S, T) = |T_A| + |S_B| + |E(S_A, T_B)|.$$

Now, $N(S_A) \cap S \subset S_B$, so $|S_B| \geq |N(S_A) \cap S|$. Also, every vertex of $N(S_A) \cap T$ corresponds to at least one edge in $E(S_A, T_B)$, so $|E(S_A, T_B)| \geq |N(S_A) \cap T|$. Thus we have:

$$\begin{aligned} \text{Cap}(S, T) &\geq |T_A| + |N(S_A) \cap S| + |N(S_A) \cap T| \\ &= |T_A| + |N(S_A)|. \end{aligned}$$

Applying Hall's Condition $|N(S_A)| \geq |S_A|$ gives:

$$\text{Cap}(S, T) \geq |T_A| + |S_A| = |A| = n.$$

Thus any cut has capacity $\geq n$, and a minimal cut has capacity exactly n , as desired. $\square$