NAME: Solution

Math 481 **Quiz 26** **Mar 31, 2025**

Proposition: If T is a tree graph, then removing a leaf v produces a smaller tree $T' = T - v$.

1. Give the definition of a tree, spelling out the terms used.

Solution: A tree T is a graph which is connected (for any two vertices there is a path in T between them) and acyclic (T contains no cycle subgraphs).

2. What is the hypothesis (setup) and the conclusion (payoff) of the above proposition?

HYPOTHESIS: T is a tree (connected and acyclic), v is a leaf ($\deg(v) = 1$), and $T' = T - v$.

CONCLUSION: T' is a tree (connected and acyclic)

3. Sketch a proof of the Proposition. Why is it true?

Informal sketch: Removing the leaf v does not break any paths in T except for those starting at v. Thus any path between vertices other than v is not broken by removing v, and lies in $T - v$, which is connected. Also, $T - v$ is clearly acyclic, since removing v from the acyclic T cannot create cycles. Therefore $T' = T - v$ is connected and acyclic, i.e. a tree.

Formal proof:

- Suppose T is a tree (connected and acyclic), with a leaf v (having $\deg(v) = 1$), and $T' = T - v$.
- In T', consider any vertices $x, y \neq v$. These are connected by a path in T:

  $$P = xv_1 \cdots v_k y \subset T.$$

  Clearly $\deg(v_i) \geq 2$, while $\deg(v) = 1$, so $v_i \neq v$ for all i, and all the vertices of P lie in T', connecting x, y inside T'. Thus T' is connected.
- Also, if $T' \subset T$ had a cycle, this would also be a cycle in T, which contradicts the hypothesis. Thus T' has no cycles.
- Since T' is connected and acyclic, it is a tree.