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Math 481 **Quiz 24 Solution** **Mar 26, 2025**

1. Define the distance between vertices in a connected graph: $\mathrm{dist}(v,w)$ means ...?

Solution: For vertices v, w in a graph G, the distance $\mathrm{dist}(v,w)$ is the minimum number of edges in a path from v to w inside G, i.e. the length of a shortest path $vPw \subset G$.

Note: We could also say "minumum number of edges in a *walk*", since the shortest walk is always a path (no repeat vertices).

2. TRUE OR FALSE: For any distinct vertices x, y, z in a connected graph,

$$\mathrm{dist}(x,z) = \mathrm{dist}(x,y) + \mathrm{dist}(y,z).$$

Solution: False. A counterexample is the complete graph K_3 with vertices $V = \{x, y, z\}$, having $\mathrm{dist}(x,y) = \mathrm{dist}(x,z) = \mathrm{dist}(y,z) = 1$, but $1 \neq 1 + 1$.

3. TRUE OR FALSE: For any distinct vertices x, y, z in a connected graph,

$$\mathrm{dist}(x,z) \leq \mathrm{dist}(x,y) + \mathrm{dist}(y,z).$$

Solution: True. Informally, a shortest path from x to y joined to a shortest path from y to z is a walk from x to z with $\mathrm{dist}(x,y) + \mathrm{dist}(y,z)$ edges, and the shortest xz-walk (or path), having $\mathrm{dist}(x,z)$ edges, cannot be longer than this.

Formal proof: Let xPy and yQz be shortest paths, by definition having length $\mathrm{dist}(x,y)$ and $\mathrm{dist}(y,z)$. Then their concatenation $xPyQz$ is a walk from x to z of length $\mathrm{dist}(x,y) + \mathrm{dist}(y,z)$. This is not necessarily a path, since Q might cross or backtrack P; however, we know that any walk can be cut down to a path with the same endpoints by skipping any loops. Thus there exists a path from x to z of length at most $\mathrm{dist}(x,y) + \mathrm{dist}(y,z)$, and a shortest path of length $\mathrm{dist}(x,z)$ cannot be longer than this, which is the conclusion to be proved.

Note: This remains true for a disconnected graph, in which some distances can be infinte:
if $\mathrm{dist}(x,z) = \infty$, then $\mathrm{dist}(x,y)$ or $\mathrm{dist}(y,z)$ must be ∞.