NAME:__________________________

Math 481 **Quiz 14** ♡ **Feb 14, 2025**

Consider the sequence defined by the recurrence:

$$a_0 = 3, \quad a_1 = 7, \quad a_k = 2a_{k-1} - a_{k-2} \text{ for } k \geq 2.$$

For example, $a_2 = 2a_1 - a_0 = 2(7) - 3 = 11$.

PROBLEM: Solve this recurrence by the Method of Generating Functions.

Step 1: Substitute the recurrence into the generating function, to write $f(x)$ as an expression involving $f(x)$. Then solve the resulting equation for $f(x)$.
Hint: Rewrite summations in dot-dot-dot notation, giving the first few terms show the pattern.

$$\begin{aligned} f(x) &= a_0 + a_1x + \sum_{k\geq 2} a_k x^k = 3 + 7x + \sum_{k\geq 2}(2a_{k-1} - a_{k-2})x^k \\ &= 3 + 7x + (2a_1x^2 - a_0x^2) + (2a_2x^3 - a_1x^3) + (2a_3x^4 - a_2x^4) + \cdots \\ &= 3 + 7x + (2a_1x^2 + 2a_2x^3 + 2a_3x^4 + \cdots) - (a_0x^2 + a_1x^3 + a_2x^4 + \cdots) \\ &= 3 + 7x + 2x(a_1x + a_2x^2 + a_3x^3 + \cdots) - x^2(a_0 + a_1x + a_2x^2 + \cdots) \\ &= 3 + 7x + 2x(f(x) - a_0) - x^2 f(x). \end{aligned}$$

Moving the $f(x)$ terms to the left side:

$$f(x) - 2xf(x) + x^2f(x) = 3 + 7x - 6x = 3 + x \quad \Longrightarrow \quad f(x) = \frac{3+x}{1-2x+x^2} = \frac{3+x}{(1-x)^2}.$$

Step 2: Find the Taylor series by adapting known series, and give an explicit formula for a_k.
Hint: Factor the denominator, distribute the numerator.)

$$\begin{aligned} f(x) &= \frac{3+x}{(1-x)^2} = (3+x)\sum_{k\geq 0}\left(\!\!\binom{2}{k}\!\!\right)x^k \\ &= \sum_{k\geq 0} 3\left(\!\!\binom{2}{k}\!\!\right)x^k + \sum_{k\geq 0}\left(\!\!\binom{2}{k}\!\!\right)x^{k+1} \\ &= 3 + \sum_{k\geq 1}\left[3\left(\!\!\binom{2}{k}\!\!\right)x^k + \left(\!\!\binom{2}{k-1}\!\!\right)\right]x^k \end{aligned}$$

Now $\left(\!\binom{2}{k}\!\right) = \binom{k+1}{k} = \binom{k+1}{1} = k+1$, so for $k \geq 1$:

$$a_k = 3\left(\!\!\binom{2}{k}\!\!\right)x^k + \left(\!\!\binom{2}{k-1}\!\!\right) = 3(k+1) + k = 4k + 3.$$

Checking: $a_0 \stackrel{\text{def}}{=} 3$,

$$a_1 \stackrel{\text{def}}{=} 7 \stackrel{!}{=} 4(1) + 1, \quad a_2 \stackrel{\text{def}}{=} 2(7) - 3 = 11 \stackrel{!}{=} 4(2) + 3, \quad a_3 \stackrel{\text{def}}{=} 2(11) - 7 = 15 \stackrel{!}{=} 4(3) + 3.$$