Math 481 **Pascal Triangle Recurrence** **Spr 2024**

Here is a model proof of a combinatorial identity using bijection (transformation).

PROPOSITION: The following identity holds for natural numbers $n \geq k \geq 1$:

$$\binom{n}{k} \;=\; \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof: To give a bijective proof, we establish that the two sides of the identity naturally count certain sets $\mathcal{A}, \mathcal{B}$, and we give an invertible mapping $T : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$.

By definition, the left side $\binom{n}{k}$ counts the k-element subsets of $[n] = \{1, 2, \ldots, n\}$:

$$\mathcal{A} \;=\; \{S \subset [n] \text{ with } |S| = k\}.$$

The right side $\binom{n-1}{k} + \binom{n-1}{k-1}$ counts subsets of $[n-1]$ with $k-1$ or k elements:

$$\mathcal{B} \;=\; \{S' \subset [n-1] \text{ with } |S'| = k \text{ or } k-1\}.$$

Define the Deletion Transform $T : \mathcal{A} \to \mathcal{B}$ by:

$$T(S) = S' = S\backslash\{n\} = \{s \in S \text{ with } s \neq n\}.$$

Note that if $n \in S$, then $|S'| = |S| - 1 = k - 1$, while if $n \notin S$, then $S' = S$ and $|S'| = k$; in either case, $S' \in \mathcal{B}$. The inverse is the Insertion Transform $T' : \mathcal{B} \to \mathcal{A}$,

$$T'(S') = \begin{cases} S' \cup \{n\} & \text{if } |S'| = k-1, \\ S' & \text{if } |S'| = k. \end{cases}$$

We check that these are inverse, undoing each other. For $n \in S \subset [n]$ we have

$$T'(T(S)) = T'(S\backslash\{n\}) = (S\backslash\{n\}) \cup \{n\} = S,$$

while for $n \notin S$ we have $T'(T(S)) = T'(S) = S$. Conversely, for $S' \subset [n-1]$ with $|S'| = k-1$, we have

$$T(T'(S')) = T(S' \cup \{n\}) = (S' \cup \{n\})\backslash\{n\} = S',$$

while for $|S'| = k$ we have $T(T'(S')) = T(S') = S'$. Thus T has inverse T'.

Therefore the Transformation Principle implies $|\mathcal{A}| = |\mathcal{B}|$, and:

$$\binom{n}{k} \;=\; |\mathcal{A}| \;=\; |\mathcal{B}| \;=\; \binom{n-1}{k-1} + \binom{n-1}{k}.$$