Math 481 # Midterm Review Problems **Oct 2021**

1. There are 100 runners in the state athletic division. Count how many possible results in each case below.

a. They all run a race, and we rank all runners in order of finishing. *Ans:* 100!

b. In the race, we rank only the top 10 runners in order, ignoring the rest. *Ans:* $100^{\underline{10}}$

c. We choose a State Team of 10 runners. *Ans:* $\binom{100}{10}$

d. Choose a State Team of 10 runners, including captain and vice-captain. *Ans:* $\binom{100}{10}10^{\underline{2}}$

e. We distribute 10 named awards (A award, B award,...) among the 100 runners, where each runner can receive multiple awards. *Ans:* 100^{10}. *Choose who gets each award.*

f. We distribute 10 named awards among the 100 runners, where each runner can receive at most one award. *Ans:* $100^{\underline{10}}$. *Equivalent to (b).*

2. We have 10 books to arrange on 4 shelves. Count how many possible arrangements in each case below.

a. 10 identical notebooks. Example: NNNNN, NN, none, NNN. *Ans:* $\left(\!\binom{4}{10}\!\right)$

b. 5 blue notebooks, 5 red notebooks. Example: BRRBR, BB, none, BRR.
Hint: Arrange 10 blank N's, then replace 5 of them by R's, the rest by B's. *Ans:* $\left(\!\binom{4}{10}\!\right)\binom{10}{5}$

c. 10 distinct novels A, B, ..., J. Example: BJFCH, DA, none, EIG
Hint: Arrange 10 blank notebooks, then replace with a sequence of novels. *Ans:* $\left(\!\binom{4}{10}\!\right)$ 10!

d. 10 novels arranged alphabetically on each shelf. Example: BCFHJ, AD, none, EGI.
Hint: Place A on a shelf, then B, etc. How many choices for each? *Ans:* 4^{10}

3. Arrange 10 novels on 4 shelves, alphabetically on each shelf, with *no empty shelves.* Example: BCFHJ, D, A, EGI. Solve this by PIE.

a. Define a set of all arrangements A, and four sets of bad arrangements $B_1, \ldots, B_4$.

b. Apply the PIE formula and compute the terms. *Ans:* $4^{10} - \binom{4}{1}3^{10} + \binom{4}{2}2^{10} - \binom{4}{3}1^{10}$

4. Problem: There are one million numbers from 0 to 999,999. How many have digit sum equal to 27? Example: The number 67,338 has sum 6+7+3+3+8 = 27.

Solve this problem by the Method of Generating Functions.

a. Step 0: Generalize to a family of counting problems with answers $a_0, a_1, a_2, \ldots$, so that the original problem is a_{27}. That is, a_k counts the number of ....

b. Step 1: Find a simple expression for $f(x) = \sum_{k\geq 0} a_k x^k$. Write an algorithm to select 6 digits, translate it into an expression for $f(x)$, and simplify using geometric series.

c. Step 2: Explicitly find the Taylor series for $f(x)$ using our knowledge of functions.
Hint: Use binomial and negative binomial series, collect all x^{27} terms, and find a_{27}.
Ans: Keep setup the same, but generalize 27 to k, so a_k counts numbers with digit sum k.

$$f(x) = (x^0 + x^1 + \cdots + x^9)^6 = \left(\frac{1-x^{10}}{1-x}\right)^6 = (1-x^{10})^6 \frac{1}{(1-x)^6}$$

$$a_{27} = \binom{6}{0}\left(\!\binom{6}{27}\!\right) - \binom{6}{1}\left(\!\binom{6}{17}\!\right) + \binom{6}{2}\left(\!\binom{6}{7}\!\right)$$