Math 481 **Hand-In Homework 1** **Jan 2026**

The Fibonacci numbers are defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ starting from $F_1 = F_2 = 1$:

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8\ F_7 = 13,\ F_8 = 21, \ldots$$

We also saw a combinatorial model of these numbers as $F_n = |D_n|$, meaning the Fibonacci number counts the set D_n of all jump-sequences:

$$d = (d_1, d_2, \ldots, d_j) \ \text{ where } \ d_i = 1 \text{ or } 2 \text{ and } d_1 + \cdots + d_j = n{-}1.$$

For example, $F_6 = 8$ counts the 8 lists of 1's and 2's summing to $6{-}1 = 5$:

$(1,1,1,1,1)$ $(1,1,1,2)$ $(1,1,2,1)$ $(1,2,1,1)$ $(2,1,1,1)$ $(1,2,2)$ $(2,1,2)$ $(2,2,1)$

Note that D_1 contains only the empty sequence $()$, and D_2 contains only (1).

1. Prove directly that the jump-sequences D_n satisfy the Fibonacci recurrence:

$$|D_n| = |D_{n-1}| + |D_{n-2}|$$

by defining an appropriate Deletion Transform bijection:

$$T : D_n \to D_{n-1} \cup D_{n-2}.$$

a. To begin proving this, write all the jump-sequences in D_6 in a row, and underneath $D_5 \cup D_4$ together in another row. You need to match these rows one-to-one by a simple, reversible transformation rule.

Hint: Delete a part of each D_6 sequence to get a smaller one, but in a reconstructible way with no information lost.

b. Give a precise definition of T for general n, specifying how a given $d \in D_n$ has a part deleted to give $d' = T(d)$ in D_{n-1} or D_{n-2}.

c. Also specify the inverse mapping $T^{-1} : D_{n-1} \cup D_{n-2} \to D_n$. How to reconstruct d from a given d' in D_{n-1} or in D_{n-2}?

2. *Proposition:* $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

a. As above, write a row of D_6, and underneath $D_1 \cup D_3 \cup D_5$ all in one row. Find a simple reversible transformation rule to match them.

Hint: Again delete a part of the D_6 sequences, but this time a part of variable size to get shorter sequences of many possible lengths.

b. Define the transformation rule for general n:

$$T : D_{2n} \to D_1 \cup D_3 \cup D_5 \cup \cdots \cup D_{2n-1}.$$

What is done to a given $d \in D_{2n}$ to get $d' = T(d)$ in one of the smaller D_i?

c. Define the inverse transformation T^{-1}, reconstructing d from d'.

d. *Extra Credit.* If you want a challenge, give a transformation proof of the following tricky identity *instead* of the above one (no need to do both):

$$\text{For even } n\text{:} \quad F_{n+1}F_{n-1} = F_n^2 + 1.$$

$$\text{For odd } n\text{:} \quad F_{n+1}F_{n-1} + 1 = F_n^2.$$

3. *Proposition:* $F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots = \sum_{i+j=n-1} \binom{j}{i}$.

Prove this with *either* of the following methods (no need to do both):

- By mathematical induction, using the defining recurrence for F_n on the left side, and the Pascal Triangle recurrence for $\binom{j}{i}$ on the right side.
- By a transformation of D_n into the sets counted by the right side: $\binom{j}{i}$ counts the i-element subsets of $\{1, \ldots, j\}$. Specify T for general n using a Position Transform, and illustrate with a table for F_6.