Math 419H **Homework 5.3(f) Solution** **Spr 2017**

Claim: There is no faithful representation of the 8-element quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ on a 3-dimensional real vector space $\mathbb{R}^3$.

Proof. Suppose we had such a representation on $\mathbb{R}^3$ with matrices given by a group homomorphism $R : Q \to \mathrm{GL}_3(\mathbb{R})$; any other basis of $\mathbb{R}^3$ would also give real matrices. The same matrices act on the complex space $\mathbb{C}^3$, and we know from earlier problems that this must split as $\mathbb{C}^3 = \mathbb{C}_\rho \oplus V_2$, a sum of a complex 1-dimensional representation and the complex irreducible 2-dimensional representation $R_2 : Q \to \mathrm{SU}_2$,

$$R_2(\pm 1) = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ R_2(\pm i) = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ R_2(\pm j) = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ R_2(\pm k) = \pm\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Since $\mathbb{C}^3 = \mathbb{R}^3 \oplus i\mathbb{R}^3$ as real vector spaces, we can define the complex conjugate of the subspace V_2, making another complex subspace $\overline{V}_2$. But since V_2 and $\overline{V}_2$ are both irreducible representations, they must be disjoint (which is impossible inside $\mathbb{C}^3$), or the same space. Thus $\overline{V}_2 = V_2$, and we have the $\mathbb{R}$-linear conjugation mapping $C : V_2 \to V_2$, $C(v) = \overline{v}$, with $C^2 = I$. Then $V_2 = V_+ \oplus V_-$ splits into ± 1 eigenspaces of C, and $V_- = iV_+$, so that $\dim_{\mathbb{R}}(V_+) = 2$. Clearly $V_+ \subset \mathbb{R}^3$, so we can find $\{v_1, v_2\} \subset \mathbb{R}^3$ with $V_+ = \mathbb{R}v_1 \oplus \mathbb{R}v_2$, and $V_2 = \mathbb{C}v_1 \oplus \mathbb{C}v_2$. With respect to this basis of $V_+ \subset \mathbb{R}^3$, the representation matrices of R_2 acting on V_2 are real: $MR_2(g)M^{-1} \in \mathrm{GL}_2(\mathbb{R})$ for a change-of-basis matrix M.

Now, $V_+ \subset \mathbb{R}^3$ has the usual dot product, and we can average this over the group Q to get an invariant positive-definite dot product on V_+. Changing to an orthonormal basis of V_+ makes the representation matrices orthogonal, and since $R_2(g)$ has determinant 1, so does the conjugated matrix: $LMR_2(g)M^{-1}L^{-1} \in \mathrm{SO}_2$. Since R_2 is faithful, we get an embedding of the non-abelian group Q into the abelian group SO_2, which is impossible.

There could not have been such a representation $\mathbb{R}^3$ to begin with.