Math 419H **Outline of Algebraic Geometry** **Spring 2017**

Geometric Object	$\Longleftrightarrow$	**Polynomial Ideal**	$\Longleftrightarrow$	**Coordinate Ring**
$V \subset \mathbb{A}^n = \mathbb{C}^n$ algebraic variety common zeroes of poly eqs	vanishing ideal $I(V)$ $\Longrightarrow$ $\Longleftarrow$ vanishing locus $V(I)$	$I \subset \mathbb{C}[x_1, \ldots, x_n]$ radical ideal $f^k \in I \Rightarrow f \in I$	quotient ring $\Longrightarrow$ $\Longleftarrow$ kernel of presentation homom $\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow R$	$R = \text{Fun}(V) = \mathbb{C}[x_1, \ldots, x_n]/I$ ring of poly functions restricted to V finitely generated $\mathbb{C}$-algebra without nilpotents: $\bar{f} \neq 0 \Rightarrow \bar{f}^k \neq 0$
EX: $V = V(x(y-x^2))$ $= \{(x, y) \in \mathbb{A}^2 \mid x(y-x^2) = 0\}$ components $V(x)$ and $V(y-x^2)$		$I = (x(y-x^2)) =$ $\{x(y-x^2)f \text{ for } f \in \mathbb{C}[x, y]\}$ principal ideal		$R = \mathbb{C}[x, y]/(x(y-x^2))$ generators $R = \mathbb{C}[\bar{x}, \bar{y}]$ with relation $\bar{x}\bar{y} = \bar{x}^3$
V irreducible $V \neq V_1 \cup V_2$ non-trivially		P prime ideal $ab \in P \Rightarrow (a \in P \text{ or } b \in P)$		$R = \text{Fun}(V)$ integral domain no zero-divisors
EX: $V = V(y-x^2)$		$I = (y-x^2)$ principal ideal of irred poly		$R = \mathbb{C}[x, y]/(y-x^2) = \mathbb{C}[\bar{x}, \bar{y}]$ with relation $\bar{y} = \bar{x}^2$
$a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ single point variety		$M_a = (x_1-a_1, \ldots, x_n-a_n) =$ $\{(x_1-a_1)f_1 + \cdots + (x_n-a_n)f_n\}$ maximal ideal		$\mathbb{C} = \text{Fun}(\text{pt})$ field
EX: intersection of varieties $V(x-2) \cap V(y-x^2) = \{(2, 4)\}$		sum of ideals $(x-2) + (y-x^2)$ $= (x-2, y-x^2) = (x-2, y-4)$ since $y-4 = (x-2)(x+2) + (y-x^2)(1)$		$R = \mathbb{C}[\bar{x}, \bar{y}] \cong \mathbb{C}$ relations $\bar{x} = 2, \bar{y} = 4$
Grothendieck scheme $\mathcal{S}$		any ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$		any finitely generated $\mathbb{C}$-algebra R
EX: tangential intersection $V(y) \cap V(y-x^2)$ $= (0, 0)$ with "infinitestimal" horizontal tangent space		sum of ideals $(y) + (y-x^2)$ $= (y, x^2)$ non-radical ideal		$R = \mathbb{C}[\bar{x}, \bar{y}] = \mathbb{C}1 \oplus \mathbb{C}\bar{x}$ relations $\bar{x}^2 = 0, \bar{y} = 0$

Hilbert Nullstellensatz

Strong: For any ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, we have $I(V(I)) = \sqrt{I} = \{f \mid f^k \in I \text{ for some } k > 0\}$, radical of I.

Weak: Every ideal $I \subsetneq \mathbb{C}[x_1, \ldots, x_n]$ is contained in some maximal ideal $M_a = (x_1-a_1, \ldots, x_n-a_n)$.
Every variety $V(I)$ for $I \neq (1)$ contains some point $a = (a_1, \ldots, a_n) \in V(I)$