Math 418H **Fall 2005**

Lecture: Mon 10/17

1. Why bother with complex numbers $\mathbb{C}$?

   - Define new number systems to solve equations that have no solutions in old number systems.
   - $x + 1 = 0$ has no soln in $\mathbb{N}$, so define $\mathbb{Z}$ (negative numbers)
   - $2x - 1 = 0$ has no soln in $\mathbb{Z}$, so define $\mathbb{Q}$ (fractions)
   - $x^2 - 2$ has no soln in $\mathbb{Q}$, so define $\mathbb{R}$ (irrational numbers)
   - $x^2 + 1 = 0$ has no soln in $\mathbb{R}$, so define $\mathbb{C}$ (imaginary numbers)

2. Formal definition of $\mathbb{C}$

   - As with $\mathbb{Q}$ and $\mathbb{R}$, we do not try to uncover the "essence" of a new number like $i = \sqrt{-1}$. We just define it by enough information to determine all its properties.
   - $\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$, pairs of real numbers: (a, b) represents the complex number $a + bi$.
   - Addition: $(a, b) + (c, d) := (a + c, b + d)$.

     Motivation: $(a{+}bi) + (c{+}di) = (a{+}c) + (b{+}d)i$.
   - Multiplication: $(a, b) \bullet (c, d) := (ac - bd, ad + bc)$.

     Motivation: $(a{+}bi) \bullet (c{+}di) = ac{+}bdi^2{+}adi{+}bci = (ac{-}bd){+}(ad{+}bc)i$.
   - Check the field axioms for $\mathbb{C}$. Identity elements: $(0, 0)$, $(1, 0)$. Multiplicative associativity:

$$\begin{aligned} [\,(a,b)\bullet(c,d)\,]\bullet(e,f) &= (ace{-}adf{-}bcf{-}bde) + (acf{+}ade{+}bce{-}bdf)i \\ &= (a,b)\bullet[\,(c,d)\bullet(e,f)\,]\,. \end{aligned}$$

   - The only tricky property is the existence of multiplicative inverses. We *should* have:

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\,.$$

     This is motivation, but proves nothing, because we have not established that $1/(a + bi)$ even exists.

- Given $(a,b) \neq (0,0)$, we *define* the multiplicative inverse as:
$$(a,b)^{-1} := \left(\frac{a}{a^2+b^2} \;,\; -\frac{b}{a^2+b^2} \right) .$$
Now we *prove* that $(a,b) \bullet (a,b)^{-1} = (1,0)$ by applying the definition of multiplication.
- Notation: a real number $a \in \mathbb{R}$ is identified with $(a,0)$, so we can regard $\mathbb{R} \subset \mathbb{C}$. Define $i := (0,1)$.
- Prove that $i^2 = -1$ and $(a,b) = a + b \bullet i$.

3. Geometric picture of $\mathbb{C}$

- Picture: $\mathbb{C} = \mathbb{R}^2$, $x + iy = (x,y)$, vectors in the real plane
- Addition of complex numbers = usual addition of vectors (diagonal of parallelogram)
- Multiplication of complex numbers = some kind of multiplication of plane vectors:
$$(a+ib) \bullet (x+iy) = (a,b) \bullet (x,y) .$$
- Multiplying by $a = (a,0)$, we have
$$a \bullet (x,y) = (ax, ay) = \text{ stretch } (x,y) \text{ by } a \,,$$
the usual scalar multiple of a vector
- Multiplying by $i = (0,1)$, we have
$$i \bullet (1,0) = (0,1) \quad , \quad i \bullet (0,1) = (-1,0)$$
and: $(x,y) \mapsto i \bullet (x,y) = (-y,x)$ is an $\mathbb{R}$-linear map. Thus:
$$i \bullet (x,y) = \text{ rotate } (x,y) \text{ by } 90^\circ .$$
- Multiplying by a unit-length vector $u = cos\theta + i \sin\theta = (\cos\theta, \sin\theta)$:
$$u \bullet (1,0) = (\cos\theta, \sin\theta) \quad , \quad u \bullet (0,1) = (-\sin\theta, \cos\theta)$$
and $(x,y) \mapsto u \bullet (x,y)$ is an $\mathbb{R}$-linear map. Thus:
$$u \bullet (x,y) = \text{ rotate } (x,y) \text{ by } \theta .$$

Lecture: Wed 10/19

1. Complex multiplication = rotation

   - For $v = (a, b) \in \mathbb{C}$, consider the multiplication map

     $$\begin{array}{rcl} M_v : \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & u \bullet (x, y) \end{array}$$

     This map is $\mathbb{R}$-linear:

     $$M_v(cx, cy) = cM_v(x, y)$$

     $$M_v(x_1 + x_2, y_1 + y_2) = M_v(x_1, y_1) + M_v(x_2, y_2) \,.$$

     for all $c \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Thus:

     $$M_v(x, y) = x\, M_v(1, 0) + y\, M_v(0, 1) \,.$$

   - Multiply by $i = (0, 1)$:

     $$i \bullet (1, 0) = (0, 1) \quad , \quad i \bullet (0, 1) = (-1, 0)$$

     $$i \bullet (x, y) = \text{ rotate } (x, y) \text{ by } 90^\circ \,.$$

   - Multiply by a unit-length vector $u = cos\theta + i \sin\theta = (\cos\theta, \sin\theta)$:

     $$u \bullet (1, 0) = (\cos\theta, \sin\theta) \quad , \quad u \bullet (0, 1) = (-\sin\theta, \cos\theta) \,.$$

     $$u \bullet (x, y) = \text{ rotate } (x, y) \text{ by } \theta \,.$$

   - Write an arbitrary vector in polar coordinates: $v = ru$, where $r \in \mathbb{R}$ and $u = cos\theta + i \sin\theta$. Then:

     $$v \bullet (x, y) = \text{ rotate } (x, y) \text{ by } \theta \text{, then stretch by } r \,.$$

2. Complex multiplication: add angles, multiply lengths

   - Consider the complex product: $v_3 = v_1 \bullet v_2$, and write each number in polar form: $v_j = r_j(\cos\theta_j + i \sin\theta_j$ for $j = 1, 2, 3$. Then:

     $$\theta_3 = \theta_1 + \theta_2 \quad , \quad r_3 = r_1 r_2 \,;$$

     that is: to multiply complex numbers, add their angles and multiply their lengths.

- *First proof:* Since the multiplcation map $(x,y) \mapsto v_j \bullet (x,y)$ is rotating by θ_j and stretching by r_j, we can describe the product $v_1 \bullet v_2 = v_1 \bullet v_2 \bullet 1$ as follows: start with unit vector 1; rotate by θ_2; stretch by r_2; rotate by θ_1; stretch by r_1. Result: rotate by $\theta_1 + \theta_2$, and stretch by $r_1 r_2$.
- *Second proof:* From the formula for complex multiplication:

$$r_1(\cos\theta_1 + i\sin\theta_1) \bullet r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \,(\,(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\,)$$

$$\overset{!}{=} r_1 r_2 \,(\,\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\,)$$

  by the angle-addition formulas.

3. Complex powers

- $2v$ is the vector v stretched by 2
- $-v$ is the vector opposite to v
- Let $v = r(\cos\theta + i\sin\theta)$.
  $v^2 = v \bullet v$ is the vector with length r^2 and angle 2θ
- $\sqrt{v}$ is a vector with length $\sqrt{r}$ and angle $\frac{1}{2}\theta$.
- There are 2 square roots because the angle θ is amiguous. We could just as well write:

$$v = r(\,\cos(\theta{+}2\pi) + i\sin(\theta{+}2\pi)\,)$$

  so that

$$\begin{aligned}\sqrt{v} &= \sqrt{r}\,(\,\cos(\tfrac{1}{2}\theta{+}\pi) + i\sin(\tfrac{1}{2}\theta{+}\pi)\,)\\ &= -\sqrt{r}\,(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta)\,.\end{aligned}$$

- *DeMoivre's Theorem:* $v^{1/n}$ is any vector with length $r^{1/n}$ and angle

$$\frac{\theta + 2k\pi}{n} = \frac{\theta}{n} + \frac{2\pi k}{n}\,.$$

  There are n such vectors evenly spaced around the circle, corresponding to the values $k = 0, 1, \ldots, n{-}1$.

4. Complex numbers as matrices

- Any linear mapping $M : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by a 2×2 matrix. If $M(1,0) = (a,b)$ and $M(0,1) = (c,d)$, then: $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, and:

$$M(x,y) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} .$$

Here we use row vectors and column vectors interchangeably: $(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}$

- The linear mapping M_u for $u = \cos\theta + i \sin\theta$ is given by the matrix:

$$M_u(x,y) = v \bullet (x,y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

This is called the *rotation matrix* of θ.

- The linear mapping M_v for $v = a + bi = ru$ is rotation by θ and stretching by r. Its matrix is:

$$M_v(x,y) = v \bullet (x,y) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

This is called a *complex multiplication matrix*.

- Consider the set of all complex mult matrices:

$$M_{\mathbf{C}} := \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ where } a, b \in \mathbb{R} \right\} .$$

This is a "copy" of the complex number field inside the ring of 2×2 matrices. That is, there is an isomorphism of fields from the complex numbers to this ring of matrices:

$$\begin{array}{rccc} \phi : & \mathbb{C} & \to & M_{\mathbf{C}} \\ & a + bi & \mapsto & \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \end{array}$$

satisfies:

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2) \quad \text{and} \quad \phi(v_1 \bullet v_2) = \phi(v_1) \cdot \phi(v_2) ,$$

where the operation on the left side of each equation is in $\mathbb{C}$, and the operation on the right side is an operation of matrices.

Math 418H **Fall 2005**

Lecture: Mon 10/24

1. Picturing complex functions

   - A complex function $f : \mathbb{C} \to \mathbb{C}\,,\ f(x+iy) = u(x,y) + i\,v(x,y)$ has real component $u(x,y)$ and imaginary component $v(x,y)$, where $u, v : \mathbb{R}^2 \to R$ are real functions on $\mathbb{C} = \mathbb{R}^2$.
   - This is the same thing as a vector field $f : \mathbb{R}^2 \to \mathbb{R}^2\,,\ f(x,y) = (u(x,y), v(x,y))$, with x-component $u(x,y)$ and y-component $v(x,y)$. This can be pictured by a field plot: draw each arrow $f(x,y)$ with its base at the point (x,y).
   - *Example 1:* The complex function $f(z) = iz$ is equivalent to the vector field: $f(x,y) = (-y, x)$ whose field plot has arrows circulating around the origin, with length proportional to their distance from the origin. This is the velocity field of a turn-table.

     For a general $\alpha = r\,\mathrm{cis}\,\theta$, the field plot of $f(z) = \alpha\,z$ is a vortex centered at the origin, with the arrows rotated by angle θ away from the outward direction, like the velocity field of water swirling down the drain.
   - *Example 2:* The complex function $f(z) = z^2 + (1+i)z + 1$ is equivalent to the vector field $f(x,y) = (x^2 - y^2 + x - y + 1\,,\, 2xy + y + x)$
   - *Example 3:* The complex function $f(z) = \overline{z}$, complex conjugate, is equivalent to the vector field $f(x,y) = (x, -y)$.

2. Derivative of a vector field

   - An arbitrary vector field $f(x,y) = (u(x,y), v(x,y))$ has a derivative matrix:
     $$Df := \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix},$$
     where
     $$u_x(x,y) = \frac{\partial u}{\partial x} := \lim_{\epsilon \to 0} \frac{u(x+\epsilon,\ y) - u(x,y)}{\epsilon}$$
     is the partial derivative of $u(x,y)$ in the x-direction, etc.
   - If $f : \mathbb{R} \to \mathbb{R}$ is an ordinary real function, its derivative $f'(a)$ gives the slope of the best linear approximation to $f(x)$ near $x = a$: for small ϵ, we have:
     $$f(a+\epsilon) \approx f(a) + f'(a)\,\epsilon\,,$$

which is just unravelling the definition of derivative:

$$f'(a) \approx \frac{f(a{+}\epsilon) - f(a)}{\epsilon}\,.$$

Similarly, for a vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$, the derivative matrix $Df(a,b)$ gives the best linear-function approximation near the point (a,b): for small (ϵ_1, ϵ_2), we have:

$$f(a{+}\epsilon_1,\, b{+}\epsilon_2) \approx f(a,b) + Df(a,b) \cdot \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix},$$

where the last operation is matrix multiplication.

- *Example 2:* For $f(x,y) = (x^2{-}y^2{+}x{-}y{+}1\,,\, 2xy{+}y{+}x)$, we have:

$$Df(x,y) = \begin{bmatrix} 2x{+}1 & 2y{+}1 \\ -2y{-}1 & 2x{+}1 \end{bmatrix}$$

- *Example 3:* For $f(x,y) = (x, -y)$, we have:

$$Df(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

3. Complex analytic functions

- We say a complex function $f(x+iy) = u(x,y) + i\,v(x,y)$ is *complex analytic* (or just *analytic*) if any of the following equivalent conditions apply.

- The partial derivatives of $f(z) = f(x+iy)$ in the real and imaginary directions are *equal*:

$$\begin{aligned} \frac{\partial f(x+iy)}{\partial x} &= \lim_{\epsilon \to 0} \frac{f(z+\epsilon) - f(z)}{\epsilon} = u_x(x,y) + i v_x(x,y) \\ \overset{!}{=} \frac{\partial f(x+iy)}{\partial iy} &= \lim_{\epsilon \to 0} \frac{f(z+i\epsilon) - f(z)}{i\epsilon} = v_y(x,y) - i u_y(x,y)\,. \end{aligned}$$

  We define the complex derivative $f'(z)$ to be the common value of these partial derivatives.

- For every value $z = x + iy$, the derivative matrix $Df(x,y)$ is a complex multiplication matrix M_{c+id} for some $c + id \in \mathbb{C}$:

$$Df := \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}.$$

We define the complex derivative $f'(z)$ to be the complex number in this multiplication matrix:

$$f'(z) := c + id = u_x + iv_x = v_y - iu_y\,.$$

- The component functions of $f(x+iy) = u(x,y) + iv(x,y)$ satisfy the *Cauchy-Riemann* partial differential equations:

$$u_x = v_y \quad , \quad v_x = -u_y\,.$$

4. Examples: analytic and non-analytic functions

- *Example 1:* $f(z) = iz\,,\ f(x,y) = (-y, x)\,,$

$$f'(z) = (u_x, v_x) = (v_y, -u_y) = (0,1) = i\,.$$

- *Example 2:* $f(z) = z^2{+}(1{+}1)z{+}1\,,\ f(x,y) = (x^2{-}y^2{+}x{-}y{+}1\,,\, 2xy{+}y{+}x)\,,$

$$f'(z) = (u_x, v_x) = (v_y, -u_y) = (2x+1, 2y+1) = 2z+1\,.$$

- *Example 3:* $f(z) = \overline{z}\,,\ f(x,y) = (x, -y)\,,$

$$f'(z) = (u_x, v_x) = (1,0) \stackrel{?}{=} (v_y, -u_y) = (-1, 0)\,.$$

  The equality does *not* hold, so $f(z)$ is *not* analytic at any z!

- For a general complex analytic $f(z)$ with roots $z = r_1, \ldots, r_n$, the field plot has a vortex around each r_i which looks approximately like the vortex of $g(z) = \alpha\, z$ for $\alpha = f'(r_i)$.

5. Combining analytic functions

- $f(z) = \alpha$ (constant function) and $f(z) = z$ are analytic
- If $f(z)$ and $g(z)$ are analytic, then:
  - $f(z) + g(z)$ is analytic and $(f(z) + g(z))' = f'(z) + g'(z)$.
  - $f(z)\, g(z)$ is analytic and $(f(z)\, g(z))' = f'(z)g(z) + f(z)g'(z)$.
  - $f(z)/g(z)$ is analytic for all z where $g(z) \neq 0$, and

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}\,.$$

- *Corollary:* All polynomial functions $f(z) \in \mathbb{C}[z]$ are complex analytic for every z. All rational functions $f(z)/g(z)$ are complex analytic except at the points z where $g(z) = 0$.

6. Fundamental Theorem of Algebra

- *Theorem:* Any polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$$

of degree $n \geq 1$ has at least one complex root $z = \alpha$ with $f(\alpha) = 0$.

- This means: the field plot of any polynomial $f(z)$ has at least one vortex. The plot of a high-degree polynomial is very complicated, so this is not at all obvious!

  Alternatively: any complex polynomial of degree n can be completely split into n linear factors:

$$f(z) = a_n(z - r_1) \cdots (z - r_n)\,.$$

  This will have fewer than n vortices if some of the r_i's coincide.

- *Strategy of Proof:* First, Cauchy's Mean Value Theorem says that for any circle in the complex plane, the value of an analytic function at the center is a certain average of the values on the circle.

- Next, Liouville's Theorem: Let $f(z)$ be complex analytic on the whole plane, with $\lim_{|z|\to\infty} f(z) = 0$, meaning that $f(z)$ becomes very small when z is far from the origin. Then $f(z)$ can only be the zero constant function: $f(z) = 0$ for all z.

  Proof: Consider any particular α, and take a very large circle centered at α. Given $\epsilon > 0$, by assumption we can take an α-centered circle large enough so that $|f(z)| < \epsilon$ for z on the circle. By Cauchy, the value $f(\alpha)$ is the average of the values $f(z)$ on the circle, so $|f(\alpha)| < \epsilon$. Since this is true for any $\epsilon > 0$, we must have $|f(\alpha)| = 0$, so $f(\alpha) = 0$. This holds for each $\alpha \in \mathbb{C}$.

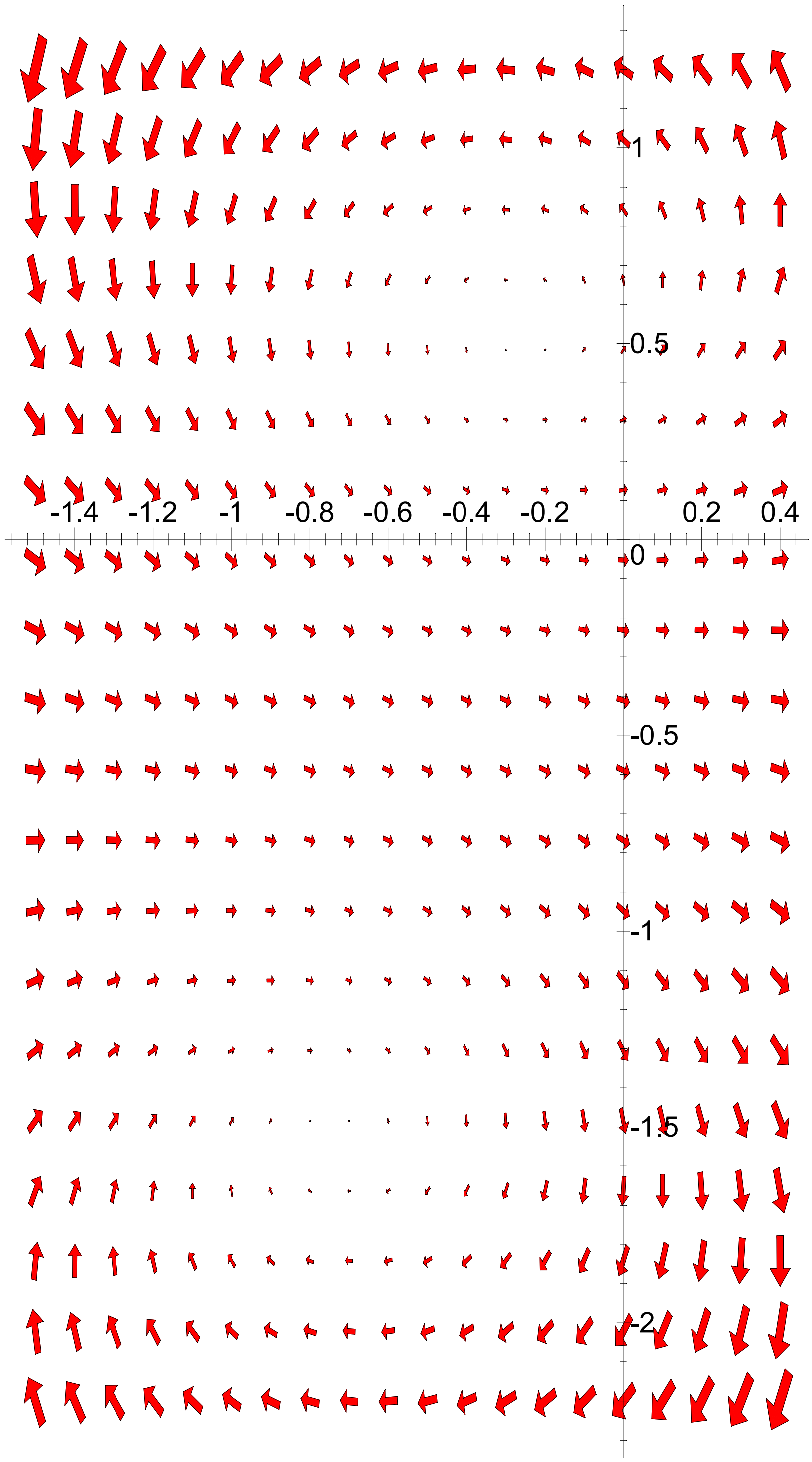
- Finally, suppose there were a polynomial function $g(z)$ with *no roots*. Then the function $f(z) = 1/g(z)$ would be analytic on the whole plane, and $|g(z)| = 1/|f(z)| \to 0$ for $|z| \to \infty$, since $\deg g(z) \geq 1$. But by Liouville, $f(z)$ can only be the zero constant function, a contradiction.

- Note that the innocent-looking *non-analytic* function:

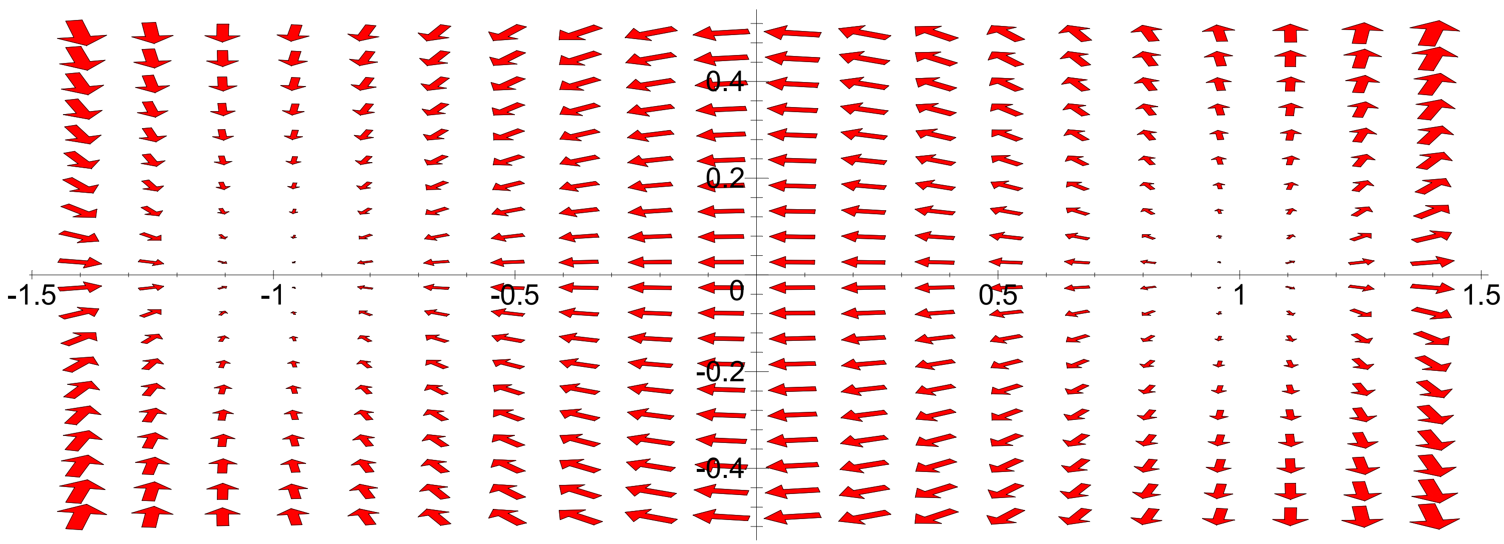
$$f(z) = z\bar{z} + 1 = |z|^2 + 1$$

  has no roots! Analytic functions are very special.

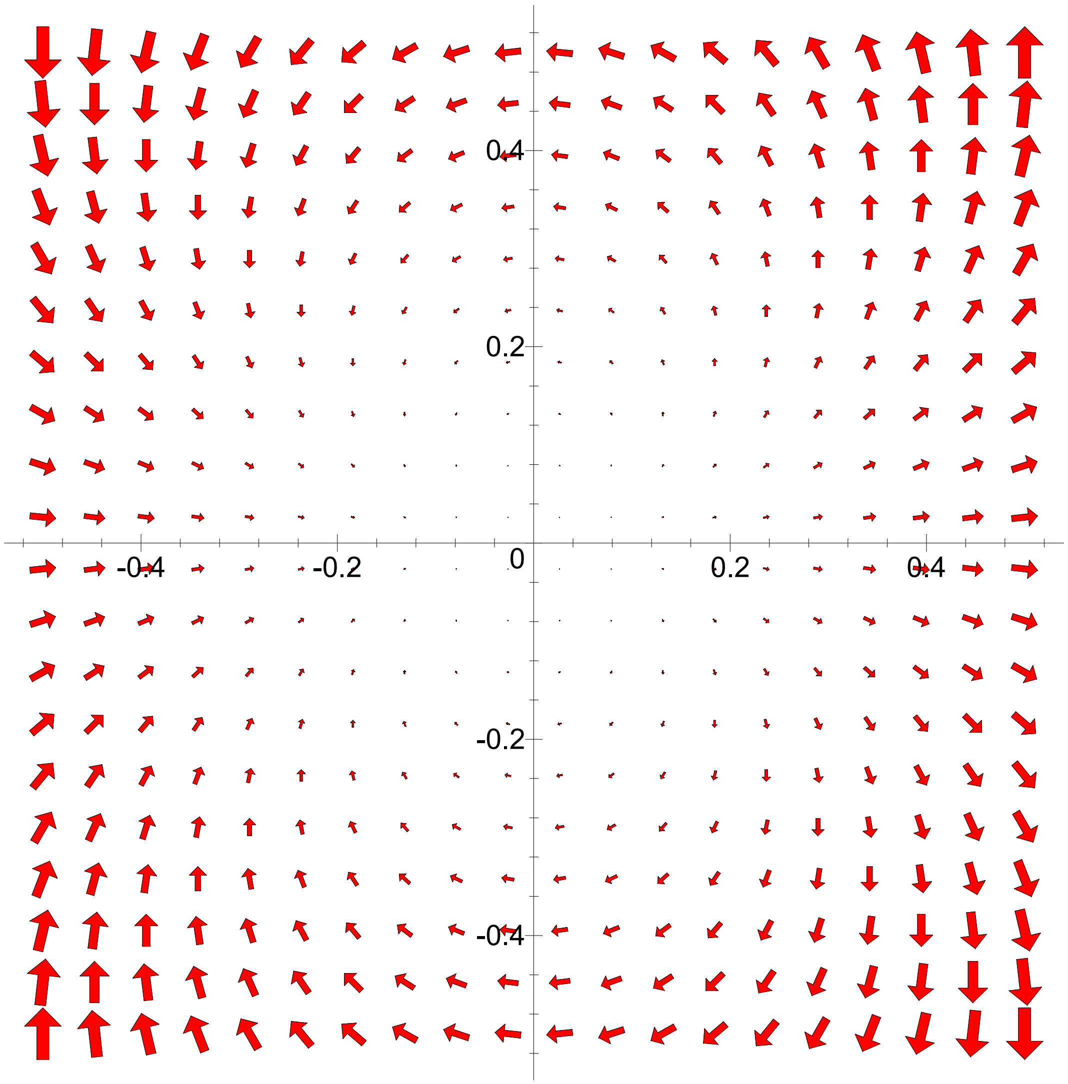
$f(z) = z^2 + (1+i)z + 1$



f(z) = z^2 - 1



f(z) = z^2



Math 418H **Fall 2005**

Lecture: Mon 10/31

1. Electromagnetic vector fields

   - Let $g(x,y) = (r(x,y), s(x,y))$ be any vector field.
   - Divergence of g measures rate of outflow from each point:

   $$\operatorname{div} g(x,y) := \frac{\partial r}{\partial x} + \frac{\partial s}{\partial y} = r_x(x,y) + s_y(x,y)\,.$$

   - Curl of g measures counter-clockwise torque (rotational force) around each point:

   $$\operatorname{curl} g(x,y) := \frac{\partial s}{\partial x} - \frac{\partial r}{\partial y} = s_x(x,y) - r_y(x,y)\,.$$

   - *An electric force field* $g(x,y)$ satisfies *Maxwell's equations*: the curl and divergence must vanish at all points:

   $$\operatorname{curl} g(x,y) = \operatorname{div} g(x,y) = 0\,.$$

   That is:

   $$\text{(Maxwell)} \qquad r_x = -s_y \quad , \quad r_y = s_x\,.$$

   These equations hold in a region with no charge present. In general, $\operatorname{div} g$ is the charge density at each point.

2. Complex analytic vs electric vector fields

   - Let $f(x+iy) = u(x,y) + i\,v(x,y)$ be complex analytic, meaning it satisfies:

   $$\text{(Cauchy-Riemann)} \qquad u_x = v_y \quad , \quad u_y = -v_x\,.$$

   - *Proposition:* Given $f(x+iy)$, let $g(x,y)$ be the complex conjugate vector field: $g(z) := \overline{f(x+iy)}$,

   $$g(x,y) := (u(x,y), -v(x,y))\,.$$

   Then clearly:

   $$f(x,y) \text{ complex analytic} \iff g(x,y) \text{ satisfies Maxwell.}$$

- *Example:* $f(z) = z\,,\quad g(x,y) = (x,-y)$. Then $f(z)$ is analytic everywhere and $\operatorname{curl} g = \operatorname{div} g = 0$.
- *Example:* $f(z) = 1/z\,,$

  $$g(x,y) = \frac{(x,y)}{x^2+y^2} = \text{ point-charge}\,,$$

  an outward force proportional to inverse of distance (which is the 2-dimensional version of Coulomb's Law). Then $f(z)$ is analytic except at the origin, and $g(x,y)$ satisfies Maxwell except at the origin, where there is a point-charge with infinite charge-density: $\operatorname{div} g(0,0) = \infty\,.$
- *Example:* $g(z) = (x,y)$ corresponds to $f(z) = \overline{z}\,.$ Then $f(z)$ is *not* analytic, and $g(x,y)$ does *not* satisfy Maxwell's equations, since $\operatorname{curl} g(x,y) = 0$ but $\operatorname{div} g(x,y) = 2$ everywhere.

3. Parametrized curves in the plane

- *Parametrized curve:* $\mathcal{C} = c(t) = (x(t), y(t))$ for $a \le t \le b$.
  We can imagine $c(t)$ as the position at time t of a particle moving along $\mathcal{C}$ from the start point $c(a) = (x(a), y(a))$ to the end point $c(b) = (x(b), y(b))\,.$ $\mathcal{C}$ is a *closed curve* if $c(a) = c(b)$.
- *Tangent vector* at point $c(t)$:

  $$c'(t) = \lim_{\epsilon \to 0} \frac{c(t+\epsilon) - c(t)}{\epsilon} = (x'(t), y'(t))\,.$$

  Rephrasing: for two points $c_0 = c(t_0)$ and $c_1 = c(t_1)$ close together along $\mathcal{C}$, the increment vector between them is approximately the velocity vector multiplied by the time increment:

  $$c_1 - c_0 \approx c'(t_1)\,(t_1 - t_0) = c'(t_1)\,\Delta t_1\,.$$

- *Example:* $\mathcal{C} = c(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$, unit circle.
  Tangent vector at $c(t)$ is: $c'(t) = (-\sin t, \cos t)\,.$
  For $t = \pi/2\,,\quad c(t) = (0,1)\,,\quad c'(t) = (-1,0)\,.$

4. Circulation around a curve

- We wish to measure the total drag or circulation of $g(x,y)$ pushing around a closed curve $\mathcal{C}\,.$ This is a large-scale version of $\operatorname{curl} g\,,$ which measures the rate of circulation of $g(x,y)$ near a particular point.

- *Drag:* The drag of a constant vector field $g(x,y) = (c,d)$ along the line segment from $(0,0)$ to (p,q) is the dot-product:

$$(c,d) \bullet (p,q) = cp + dq\,,$$

the product of vector lengths times cos of the angle between.

- *Circulation line integral of* $g(x,y)$ along $\mathcal{C}$. Mark N points of $\mathcal{C}$:

$$c_0, c_1, \ldots, c_N = c_0\,,$$

with $c_j = c(t_j)$. We have:

$$c_j - c_{j-1} \approx c'(t_j)\,(t_j - t_{j-1}) = c'(t_j)\,\Delta t_j\,.$$

We can compute the total circulation of $g(x,y)$ around $\mathcal{C}$ by adding up the drag along each tiny line segment from c_{j-1} to c_j:

$$\begin{aligned} \oint_{\mathcal{C}} g(x,y) \bullet dc \quad &:= \quad \lim_{N\to\infty} \sum_{j=1}^{N} g(c_j) \bullet (c_j - c_{j-1}) \\ &:= \quad \lim_{N\to\infty} \sum_{j=1}^{N} g(c(t_j)) \bullet c'(t_j)\ \Delta t_j \\ &= \quad \int_{t=a}^{b} g(c(t)) \bullet c'(t)\,dt\,. \end{aligned}$$

Note that $g(c(t)) \bullet c'(t)$ is a scalar-valued function of t, so the last line is an ordinary integral.

- *Example:* Let $\mathcal{C} = (\cos t, \sin t)$ for $0 \le t \le 2\pi$, and $g(x,y) = (1,0)$ a horizontal constant vector field. Since the drag on top of the curve cancels the opposite drag on the bottom, we expect zero circulation. In fact:

$$\begin{aligned} \oint_{\mathcal{C}} g(c) \bullet dc \quad &= \quad \int_{t=0}^{2\pi} g(\cos t, \sin t) \bullet (\cos' t, \sin' t)\,dt \\ &= \quad \int_{t=0}^{2\pi} (1,0) \bullet (-\sin t, \cos t)\,dt = \int_{t=0}^{2\pi} -\sin t\ dt = 0\,. \end{aligned}$$

5. Global outflow via line integrals

- We wish to measure the total outflow or flux of $g(x,y)$ across a closed curve $\mathcal{C}$. This is a large-scale version of $\operatorname{div} g(x,y)$, which measures the rate of outflow near a particular point.

- *Flux:* The flow of a constant vector field $g(x,y) = (c,d)$ across a line segment from $(0,0)$ to (p,q) is the cross-product:

$$(c,d)\times(p,q) = cq - dp\,,$$

the product of vector lengths times sin of the angle between.

- *Flux line integral of* $g(x,y)$ along $\mathcal{C}$. As before, we compute the total outflow as:

$$\oint_{\mathcal{C}} g(x,y)\times dc = \lim_{N\to\infty}\sum_{j=1}^{N} g(c_j)\times(c_j - c_{j-1}) = \int_{t=a}^{b} g(c(t))\times c'(t)\,dt\,.$$

- *Example:* Again let $\mathcal{C} = (\cos t, \sin t)$ and $g(x,y) = (1,0)$. Since inflow on the left should cancel outflow on the right, we expect zero flux. In fact:

$$\oint_{\mathcal{C}} g(c)\times dc = \int_{t=0}^{2\pi} (1,0)\times(-\sin t, \cos t)\,dt = \int_{t=0}^{2\pi} \cos t\ dt = 0\,.$$

6. Green's Theorems: global versus local

- Let R be a region on the plane whose boundary is a simple closed curve $\mathcal{C}$ (oriented counter-clockwise). Let $g(x,y)$ be vector field which is defined and differentiable at every point of R.
- *Theorem:* The circulation of g around the boundary curve is equal to the total curl of g inside the region:

$$\oint_{\mathcal{C}} g(c)\bullet dc = \iint_R \operatorname{curl} g(x,y)\ dx\,dy\,,$$

where the right side is a double integral over the region R.

- *Theorem:* The flux of g around the boundary curve is equal to the total divergence of g inside the region:

$$\oint_{\mathcal{C}} g(c)\times dc = \iint_R \operatorname{div} g(x,y)\ dx\,dy\,.$$

- *Proof:* Divide R into little regions, and write the total line integral as a sum of line integrals over tiny regions. Inside each tiny region, $g(x,y)$ can be replaced by its linear approximation, so that we can compute the tiny line integrals to be the area times $\operatorname{curl} g$ or $\operatorname{div} g$.
- *Corollary:* If $g(x,y)$ is an electical force field with $\operatorname{curl} g = \operatorname{div} g = 0$ inside the region R, then g has zero circulation and flux over the boundary curve $\mathcal{C}$:

$$\oint_{\mathcal{C}} g(c)\bullet dc = \oint_{\mathcal{C}} g(c)\times dc = 0$$

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Lecture: Wed 11/2

1. Complex line integral

   - Given a complex derivative $F'(z)$, we would like to recover the orginal function $F(z)$ by integrating. This is done as follows: let $\mathcal{C}$ be a non-closed curve with start-point $\alpha = c(a)$ and end-point $\beta = c(b)$. Mark N points $\alpha = c_0, c_1, \ldots, c_N = \beta$, with $c_j = c(t_j)$. Then:

$$\begin{aligned} F(\beta) - F(\alpha) &= \lim_{N\to\infty} \sum_{j=1}^{N} F(c_j) - F(c_{j-1}) \\ &= \lim_{N\to\infty} \sum_{j=1}^{N} \frac{F(c_j)-F(c_{j-1})}{c_j - c_{j-1}} \; \frac{c_j - c_{j-1}}{\Delta t_j} \; \Delta t_j \\ &= \int_{t=a}^{b} F'(c(t))\, c'(t)\, dt \end{aligned}$$

   - Thus, the correct integral to use is the *complex line integral:*

$$\oint_{\mathcal{C}} f(z)\, dz := \int_{t=a}^{b} f(c(t))\, c'(t)\, dt\,,$$

   where the product in the integral is complex multiplication, and the result is a complex number. That is, if $f(x+iy) = u(x,y) + i\, v(x,y)$ and $c(t) = x(t) + i\, y(t)$, then:

$$\begin{aligned} \oint_{\mathcal{C}} f(z)\, dz &= \int_{t=a}^{b} u(c(t))\, x'(t) - v(c(t))\, y'(t) \; dt \\ &\quad + i \int_{t=a}^{b} u(c(t))\, y'(t) + v(c(t))\, x'(t) \; dt \end{aligned}$$

   - *Fundamental Theorem of Calculus*: If $F(z)$ is analytic, and $\mathcal{C}$ is a (not necessarily closed) curve from α to β, then:

$$F(\beta) - F(\alpha) = \oint_{\mathcal{C}} F'(z)\, dz\,.$$

- *Example:* $f(z) = 1/z$, $\mathcal{C} = c(t) = (r\cos t\,,\, r\sin t)$. Then $f(x{+}iy) = (x{-}iy)/(x^2{+}y^2)$, and:

$$\begin{aligned}
\oint_{\mathcal{C}} f(z)\,dz &= \int_{t=0}^{2\pi} f(r\cos t + ir\sin t)\,(r\cos' t + ir\sin' t)\,dt \\
&= \int_{t=0}^{2\pi} \frac{1}{r^2}(r\cos t - ir\sin t)\,(-r\sin t + ir\cos t)\,dt \\
&= \int_{t=0}^{2\pi} i(\cos^2 t + \sin^2 t)\,dt = 2\pi i
\end{aligned}$$

2. Cauchy Integral Theorem

- *Theorem:* If R is a plane region whose boundary is the closed curve $\mathcal{C}$, and $f(z)$ is complex analytic for every $z \in R$, then the complex line integral of $f(z)$ over $\mathcal{C}$ is zero:

$$\oint f(z)\,dz = 0\,.$$

- *First Proof:* If we can find $F(z)$ with $f(z) = F'(z)$, and we take $\alpha = \beta$ being the start- and end-point of the closed curve $\mathcal{C}$, then:

$$\int_{\mathcal{C}} f(z)\,dz = F(\alpha) - F(\beta) = 0\,.$$

For example, if $f(z) = z^2{+}1$ then we can take $F(z) = \frac{1}{3}z^3{+}z$. But how do we find such an $F(z)$ in general? For example, $f(z) = 1/z$ does *not* satisfy the Theorem, so it *cannot* be the derivative of any function $F(z)$. We need a better proof.

- *Second Proof:* We reduce the complex line integral of $f(z)$ to circulation and flux integrals of the corresponding electric field, the conjugate $g(z) := \overline{f(z)}$ with $\operatorname{curl} g = \operatorname{div} g = 0$. First, note that the complex product relates to the dot and cross products as follows:

$$\alpha\,\beta = \overline{\alpha} \bullet \beta + \overline{\alpha} \times \beta\,.$$

(Just write out real and imaginary parts of both sides.) Thus:

$$\begin{aligned}
\oint_{\mathcal{C}} f(z)\,dz &= \int_{t=a}^{b} f(c(t))\,c'(t)\,dt \\
&= \int_{t=a}^{b} \overline{f(c(t))} \bullet c'(t)\,dt + i\int_{t=a}^{b} \overline{f(c(t))} \times c'(t)\,dt \\
&= \oint_{\mathcal{C}} g(c) \bullet dc + i\oint_{\mathcal{C}} g(c) \times dc \\
&= \iint_R \operatorname{curl} g(x,y)\,dx\,dy + i\iint_R \operatorname{div} g(x,y)\,dx\,dy = 0
\end{aligned}$$

3. Cauchy Mean Value Theorem

- Let $\mathcal{C} = \mathcal{C}(r, \gamma)$ be a circle with radius r and center γ, and suppose $f(z)$ is complex analytic in the disk bounded by $\mathcal{C}$. Then the average value of $f(z)$ on the circle $\mathcal{C}$ is equal to the value $f(\gamma)$ in the center:

$$\frac{1}{2\pi r} \int_{\mathcal{C}(r,\gamma)} f(c(t))\, |c'(t))|\, dt = f(\gamma)\,.$$

- *Proof:* First, note that $c'(t) = i\,(c(t){-}\gamma)$, so:

$$\begin{aligned} \oint_{\mathcal{C}} \frac{f(z)}{z{-}\gamma}\, dz &= \int_{t=0}^{2\pi} \frac{f(c(t))}{c(t){-}\gamma}\, c'(t)\, dt \\ &= i \int_{t=0}^{2\pi} f(c(t))\, dt \\ &= \frac{i}{r} \int_{t=0}^{2\pi} f(c(t))\, |c'(t)|\, dt\,. \end{aligned}$$

Thus, the average value can be computed as:

$$A(r) := \frac{1}{2\pi r} \int_{\mathcal{C}(r,\gamma)} f(c(t))\, |c'(t))|\, dt = \frac{1}{2\pi i} \oint_{\mathcal{C}(r,\gamma)} \frac{f(z)}{z{-}\gamma}\, dz$$

Let $\mathcal{D}$ be the closed curve which first rounds the circle $\mathcal{C}(r, \gamma)$ counterclockwise, then traverses a radial line segment from radius r to a smaller radius ϵ, then rounds the circle $\mathcal{C}(\epsilon, \gamma)$ clockwise, then goes back along the same radis from ϵ to r.

The closed curve $\mathcal{D}$ is the boundary of a ring-shaped region in which $f(z)/(z{-}\gamma)$ is analytic, so that the complex integral vanishes by Cauchy's Theorem:

$$0 = \oint_{\mathcal{D}} \frac{f(z)}{z{-}\gamma}\, dz = \oint_{\mathcal{C}(r,\gamma)} \frac{f(z)}{z{-}\gamma}\, dz - \oint_{\mathcal{C}(\epsilon,\gamma)} \frac{f(z)}{z{-}\gamma}\, dz\,. = A(r) - A(\epsilon)\,.$$

That is, the average does not depend on the radius of the circle. But $f(z)$ is continuous, so as the circle $\mathcal{C}(r, \epsilon)$ approaches the central point γ, the average value of $f(z)$ on the circle approaches $f(\gamma)$:

$$A(r) = A(\epsilon) = \lim_{\epsilon \longrightarrow 0} A(\epsilon) = f(\gamma)\,.$$

Lecture: Mon 11/7

1. Fundamental Theorem of Algebra

- *Theorem:* Any polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$$

of degree $n \geq 1$ has at least one complex root $z = \alpha$ with $f(\alpha) = 0$.

- *First step:* We give a proof by contradiction. Suppose $f(z)$ were a polynomial with *no roots*. Then its reciprocal $g(z) := 1/f(z)$ would be analytic everywhere. Furthermore:

$$\lim_{|z|\to\infty} |f(z)| = \lim_{|z|\to\infty} |a_n z^n| = \infty\,,$$

meaning that $f(z)$ has large radius if z is far from the origin. Thus $\lim_{|z|\to\infty} g(z) = 0$, meaning that $g(z)$ has small radius when z is far from the origin.

- *Second step, Liouville's Theorem:* Let $g(z)$ be a function which is complex analytic on the whole plane, with $\lim_{|z|\to\infty} g(z) = 0$. Then $g(z)$ can only be the zero constant function: $g(z) = 0$ for all z.

Proof: Consider any particular $\gamma \in \mathbb{C}$, and take a very large circle $\mathcal{C}(r,\gamma)$ with radius r and center γ. Given $\epsilon > 0$, by assumption we can take radius large enough so that $|g(z)| < \epsilon$ for z on the circle $\mathcal{C}(r,\gamma)$. By Cauchy's Mean Value Theorem, the value $g(\gamma)$ at the center is the average of the value of $g(z)$ on the circle $\mathbb{C}(r,\gamma) = c(t)$:

$$g(\gamma) = \operatorname*{Avg}_{c\in\mathcal{C}(r,\gamma)} g(c) := \frac{1}{2\pi r}\int g(c(t))\ |c'(t)|\ dt\,.$$

Taking lengths and applying the triangle inequality,

$$|\operatorname*{Avg}_{\mathcal{C}} F(c)\,| \leq \operatorname*{Avg}_{\mathcal{C}} |F(c)|\,,$$

we have:

$$|g(\gamma)| = |\operatorname*{Avg}_{\mathcal{C}(r,\gamma)} g(c)\,| \leq \operatorname*{Avg}_{\mathcal{C}(r,\gamma)} |g(c)| \leq \epsilon\,.$$

Since this is true for any $\epsilon > 0$, we must have $|g(\gamma)| = 0$. This holds for each $\gamma \in \mathbb{C}$.

- *Third step:* Since $g(z) = 1/f(z)$ is not the zero constant function, we have a contradiction. Thus there cannot exist any non-vanishing polynomial $f(z) \in \mathbb{C}[z]$.

- Paraphrasing: Liouville's Theorem says that if a non-constant analytic function becomes very small as $|z| \to \infty$, then $g(z)$ must compensate for this by having non-anlytic points somewhere (for example, blowing up to infinity). Hence, if an analytic $f(z)$ becomes very large as $|z| \to \infty$ (as does a polynomial), then $f(z)$ must compensate for this by vanishing somewhere, i.e., having roots.
- This is a pure existence proof: it shows that a root-free polynomial function $f(z)$ would lead to an analytic function $g(z)$ violating the Cauchy Mean Value Theorem. The proof gives no clue how to find a root for a given $f(z)$: we will give an algorithm for this next time.

2. Factoring polynomials

- *Proposition:* Every monic complex polynomial $f(z)$ of degree n can be uniquely factored in $\mathbb{C}[z]$ as a product of n linear functions.

$$f(z) = (z-\alpha_1) \cdots (z-\alpha_n)\,.$$

That is, the irreducible polynomials of $\mathbb{C}[z]$ are linear.

Proof: By the Fundamental Theorem, $f(z)$ has a root $z = \alpha$ and thus a linear factor: $f(z) = (z-\alpha_1)\, f_1(z)$, where $f_1(z)$ has degree $n-1$. Repeat this for $f_1(z)$ until all factors are linear.

- *Proposition:* Every monic real polynomial $f(z)$ of degree n can be uniquely factored in $\mathbb{R}[z]$ as a product of linear and quadratic functions:

$$f(z) = (z-\alpha_1) \cdots (z-\alpha_k)\, q_1(z) \cdots q_\ell(z)\,,$$

where $\alpha_j \in \mathbb{R}$, $q_j(z) \in \mathbb{R}[z]$ has degree 2, and $k + 2\ell = n$. That is, the irreducible polynomials of $\mathbb{R}[z]$ are linear and quadratic.

Proof: A real polynomial $f(z) \in \mathbb{R}[z]$ can be factored into complex linear factors as above. But if $f(\alpha) = 0$, then $f(\overline{\alpha}) = \overline{f(\alpha)} = 0$, so the non-real roots come in complex conjugate pairs. Each such pair $\alpha \neq \overline{\alpha}$ with $\alpha = a + bi$ gives a real factor:

$$(z-\alpha)\,(z-\overline{\alpha}) = z^2 + (\alpha+\overline{\alpha})\,z + \alpha\overline{\alpha} = z^2 + 2az + (a^2+b^2) \in \mathbb{R}[z]\,.$$

These factors are irreducible in $\mathbb{R}[z]$ since their roots $\alpha, \overline{\alpha}$ are not in $\mathbb{R}$ by assumption.

- *Example:* Let $f(z) = z^4 + 1$, having roots $\alpha_1 = \mathrm{cis}(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}(1+i)$, $\alpha_2 = \mathrm{cis}(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}(-1+i)$, and their conjugates $\overline{\alpha}_1$, $\overline{\alpha}_2$. Factoring:

$$\begin{aligned} f(z) &= (z-\alpha_1)(z-\overline{\alpha}_1)(z-\alpha_2)(z-\overline{\alpha}_2) \\ &= (z^2 + (\alpha_1+\overline{\alpha}_1)\,z + \alpha_1\overline{\alpha}_1)\ (z^2 + (\alpha_2+\overline{\alpha}_2)\,z + \alpha_2\overline{\alpha}_2) \\ &= (z^2 + \sqrt{2}\,z + 1)\ (z^2 - \sqrt{2}\,z + 1) \end{aligned}$$