Math 418H **Lecture: Wed 11/23** **Fall 2005**

1. Dihedral group D_n

   - $D_n := \mathrm{Sym}(X)$ where $X \subset \mathbb{R}^2$ is a regular n-gon in the plane, with its rigid structure. That is, a symmetry $\pi : X \to X$ must preserve distance: $\mathrm{dist}(\pi(x), \pi(y)) = \mathrm{dist}(x, y)$ for each pair of points $x, y \in X$.
   - A symmetry $\pi : X \to X$ must permute the n vertices, and is determined by this permutation. Thus we may consider the dihedral group as a subgroup of the symmetric group (all permutations): $D_n \subset S_n$.
   - A symmetry of the n-gon must take adjacent vertices to adjacent vertices. Thus, there are n choices for $\pi(1)$, but only 2 choices for $\pi(2) = \pi(1) \pm 1$, since $\pi(2)$ must be one of the vertices adjacent to $\pi(1)$. Furthermore, $\pi(3)$ must be the unique remaining vertex adjacent to $\pi(2)$, etc. Thus, the number of symmetries is: $|D_n| = 2n$.
   - Consider the reflection and the rotation:

     $$\alpha := \begin{pmatrix} 1 & 2 & \cdots & n{-}1 & n \\ n & n{-}1 & \cdots & 2 & 1 \end{pmatrix} \qquad \beta := \begin{pmatrix} 1 & 2 & \cdots & n{-}1 & n \\ 2 & 3{-}1 & \cdots & n & 1 \end{pmatrix}.$$

     We have the identity ι, the $n{-}1$ rotation symmetries $\beta, \beta^2, \ldots, \beta^{n-1}$, and we may check that $\alpha, \alpha\beta, \alpha\beta^2, , \ldots, \alpha\beta^{n-1}$ are n distinct reflection symmetries. Thus we have listed all $2n$ elements:

     $$D_n = \left\{ \begin{matrix} \iota, & \beta, & \beta^2, & \ldots, & \beta^{n-1} \\ \alpha, & \alpha\beta, & \alpha\beta^2, & \ldots, & \alpha\beta^{n-1} \end{matrix} \right\}$$

   - The relations:

     $$\alpha^2 = \beta^n = \iota \quad , \quad \beta\alpha = \alpha\beta^{n-1}$$

     allow us to multiply arbitrary expressions of the form $\alpha^i\beta^j$. Rewrite this as $\beta\alpha = \alpha\beta^{-1}$, so that:

     $$\begin{aligned} \beta^j \cdot \beta^k = \beta^{j+k \bmod n} \quad &, \quad \beta^j \cdot \alpha\beta^k = \alpha\beta^{-j+k \bmod n} \\ \alpha\beta^j \cdot \beta^k = \alpha\beta^{j+k \bmod n} \quad &, \quad \alpha\beta^j \cdot \alpha\beta^k = \beta^{-j+k \bmod n}. \end{aligned}$$

2. Rotation vs reflection symmetries

   - Cyclic group $C_n = \{\iota, \beta, \beta^2, \ldots, \beta^{n-1}\}$ is a group generated by one element β with the relation $\beta^n = \iota$, and multiplication $\beta^j\beta^k = \beta^{j+k \bmod n}$.
   - Clearly $C_n \subset D_n$ is a subgroup. It should thus correspond to the symmetries of an n-gon with *decorations*, i.e. extra structure which decreases the number of symmetries. Indeed: $C_n = \mathrm{Sym}(X)$ where X is an n-gon with an arrow drawn on each edge.