Lecture: Wed 11/2

1. Complex line integral

   - Given a complex derivative $F'(z)$, we would like to recover the orginal function $F(z)$ by integrating. This is done as follows: let $\mathcal{C}$ be a non-closed curve with start-point $\alpha = c(a)$ and end-point $\beta = c(b)$. Mark N points $\alpha = c_0, c_1, \dots, c_N = \beta$, with $c_j = c(t_j)$. Then:

$$\begin{aligned} F(\beta) - F(\alpha) &= \lim_{N\to\infty} \sum_{j=1}^{N} F(c_j) - F(c_{j-1}) \\ &= \lim_{N\to\infty} \sum_{j=1}^{N} \frac{F(c_j) - F(c_{j-1})}{c_j - c_{j-1}} \; \frac{c_j - c_{j-1}}{\Delta t_j} \; \Delta t_j \\ &= \int_{t=a}^{b} F'(c(t))\, c'(t)\, dt \end{aligned}$$

   - Thus, the correct integral to use is the *complex line integral:*

$$\oint_{\mathcal{C}} f(z)\, dz := \int_{t=a}^{b} f(c(t))\, c'(t)\, dt\,,$$

   where the product in the integral is complex multiplication, and the result is a complex number. That is, if $f(x+iy) = u(x,y) + i\,v(x,y)$ and $c(t) = x(t) + i\,y(t)$, then:

$$\begin{aligned} \oint_{\mathcal{C}} f(z)\, dz &= \int_{t=a}^{b} u(c(t))\, x'(t) - v(c(t))\, y'(t)\; dt \\ &\quad + i \int_{t=a}^{b} u(c(t))\, y'(t) + v(c(t))\, x'(t)\; dt \end{aligned}$$

   - *Fundamental Theorem of Calculus*: If $F(z)$ is analytic, and $\mathcal{C}$ is a (not necessarily closed) curve from α to β, then:

$$F(\beta) - F(\alpha) = \oint_{\mathcal{C}} F'(z)\, dz\,.$$

- *Example:* $f(z) = 1/z$, $\mathcal{C} = c(t) = (r\cos t\,,\, r\sin t)$. Then $f(x{+}iy) = (x{-}iy)/(x^2{+}y^2)$, and:

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,dz &= \int_{t=0}^{2\pi} f(r\cos t + ir\sin t)\,(r\cos' t + ir\sin' t)\,dt \\ &= \int_{t=0}^{2\pi} \frac{1}{r^2}(r\cos t - ir\sin t)\,(-r\sin t + ir\cos t)\,dt \\ &= \int_{t=0}^{2\pi} i(\cos^2 t + \sin^2 t)\,dt = 2\pi i\end{aligned}$$

2. Cauchy Integral Theorem

- *Theorem:* If R is a plane region whose boundary is the closed curve $\mathcal{C}$, and $f(z)$ is complex analytic for every $z \in R$, then the complex line integral of $f(z)$ over $\mathcal{C}$ is zero:

$$\oint f(z)\,dz = 0\,.$$

- *First Proof:* If we can find $F(z)$ with $f(z) = F'(z)$, and we take $\alpha = \beta$ being the start- and end-point of the closed curve $\mathcal{C}$, then:

$$\int_{\mathcal{C}} f(z)\,dz = F(\alpha) - F(\beta) = 0\,.$$

For example, if $f(z) = z^2{+}1$ then we can take $F(z) = \frac{1}{3}z^3{+}z$. But how do we find such an $F(z)$ in general? For example, $f(z) = 1/z$ does *not* satisfy the Theorem, so it *cannot* be the derivative of any function $F(z)$. We need a better proof.

- *Second Proof:* We reduce the complex line integral of $f(z)$ to circulation and flux integrals of the corresponding electric field, the conjugate $g(z) := \overline{f(z)}$ with $\operatorname{curl} g = \operatorname{div} g = 0$. First, note that the complex product relates to the dot and cross products as follows:

$$\alpha\,\beta = \overline{\alpha}\bullet\beta + \overline{\alpha}\times\beta\,.$$

(Just write out real and imaginary parts of both sides.) Thus:

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,dz &= \int_{t=a}^{b} f(c(t))\,c'(t)\,dt \\ &= \int_{t=a}^{b} \overline{f(c(t))}\bullet c'(t)\,dt + i\int_{t=a}^{b} \overline{f(c(t))}\times c'(t)\,dt \\ &= \oint_{\mathcal{C}} g(c)\bullet dc + i\oint_{\mathcal{C}} g(c)\times dc \\ &= \iint_R \operatorname{curl} g(x,y)\,dx\,dy + i\iint_R \operatorname{div} g(x,y)\,dx\,dy = 0\end{aligned}$$

3. Cauchy Mean Value Theorem

- Let $\mathcal{C} = \mathcal{C}(r, \gamma)$ be a circle with radius r and center γ, and suppose $f(z)$ is complex analytic in the disk bounded by $\mathcal{C}$. Then the average value of $f(z)$ on the circle $\mathcal{C}$ is equal to the value $f(\gamma)$ in the center:

$$\frac{1}{2\pi r} \int_{\mathcal{C}(r,\gamma)} f(c(t))\, |c'(t))|\, dt = f(\gamma)\,.$$

- *Proof:* First, note that $c'(t) = i\,(c(t) - \gamma)$, so:

$$\begin{aligned} \oint_{\mathcal{C}} \frac{f(z)}{z-\gamma}\, dz &= \int_{t=0}^{2\pi} \frac{f(c(t))}{c(t)-\gamma}\, c'(t)\, dt \\ &= i \int_{t=0}^{2\pi} f(c(t))\, dt \\ &= \frac{i}{r} \int_{t=0}^{2\pi} f(c(t))\, |c'(t)|\, dt\,. \end{aligned}$$

Thus, the average value can be computed as:

$$A(r) := \frac{1}{2\pi r} \int_{\mathcal{C}(r,\gamma)} f(c(t))\, |c'(t))|\, dt = \frac{1}{2\pi i} \oint_{\mathcal{C}(r,\gamma)} \frac{f(z)}{z-\gamma}\, dz$$

Let $\mathcal{D}$ be the closed curve which first rounds the circle $\mathcal{C}(r, \gamma)$ counterclockwise, then traverses a radial line segment from radius r to a smaller radius ϵ, then rounds the circle $\mathcal{C}(\epsilon, \gamma)$ clockwise, then goes back along the same radis from ϵ to r.

The closed curve $\mathcal{D}$ is the boundary of a ring-shaped region in which $f(z)/(z-\gamma)$ is analytic, so that the complex integral vanishes by Cauchy's Theorem:

$$0 = \oint_{\mathcal{D}} \frac{f(z)}{z-\gamma}\, dz = \oint_{\mathcal{C}(r,\gamma)} \frac{f(z)}{z-\gamma}\, dz - \oint_{\mathcal{C}(\epsilon,\gamma)} \frac{f(z)}{z-\gamma}\, dz\,. = A(r) - A(\epsilon)\,.$$

That is, the average does not depend on the radius of the circle. But $f(z)$ is continuous, so as the circle $\mathcal{C}(r, \epsilon)$ approaches the central point γ, the average value of $f(z)$ on the circle approaches $f(\gamma)$:

$$A(r) = A(\epsilon) = \lim_{\epsilon \to 0} A(\epsilon) = f(\gamma)\,.$$