Math 418H **Fall 2005**

Lecture: Mon 10/31

1. Electromagnetic vector fields

   - Let $g(x,y) = (r(x,y), s(x,y))$ be any vector field.
   - Divergence of g measures rate of outflow from each point:

   $$\operatorname{div} g(x,y) := \frac{\partial r}{\partial x} + \frac{\partial s}{\partial y} = r_x(x,y) + s_y(x,y)\,.$$

   - Curl of g measures counter-clockwise torque (rotational force) around each point:

   $$\operatorname{curl} g(x,y) := \frac{\partial s}{\partial x} - \frac{\partial r}{\partial y} = s_x(x,y) - r_y(x,y)\,.$$

   - *An electric force field* $g(x,y)$ satisfies *Maxwell's equations*: the curl and divergence must vanish at all points:

   $$\operatorname{curl} g(x,y) = \operatorname{div} g(x,y) = 0\,.$$

   That is:

   $$\text{(Maxwell)} \qquad r_x = -s_y \quad , \quad r_y = s_x\,.$$

   These equations hold in a region with no charge present. In general, $\operatorname{div} g$ is the charge density at each point.

2. Complex analytic vs electric vector fields

   - Let $f(x+iy) = u(x,y) + i\,v(x,y)$ be complex analytic, meaning it satisfies:

   $$\text{(Cauchy-Riemann)} \qquad u_x = v_y \quad , \quad u_y = -v_x\,.$$

   - *Proposition:* Given $f(x+iy)$, let $g(x,y)$ be the complex conjugate vector field: $g(z) := \overline{f(x+iy)}$,

   $$g(x,y) := (u(x,y), -v(x,y))\,.$$

   Then clearly:

   $$f(x,y) \text{ complex analytic} \iff g(x,y) \text{ satisfies Maxwell.}$$

- *Example:* $f(z) = z\,, \quad g(x,y) = (x,-y)$. Then $f(z)$ is analytic everywhere and $\operatorname{curl} g = \operatorname{div} g = 0$.
- *Example:* $f(z) = 1/z\,,$

  $$g(x,y) = \frac{(x,y)}{x^2+y^2} = \text{ point-charge}\,,$$

  an outward force proportional to inverse of distance (which is the 2-dimensional version of Coulomb's Law). Then $f(z)$ is analytic except at the origin, and $g(x,y)$ satisfies Maxwell except at the origin, where there is a point-charge with infinite charge-density: $\operatorname{div} g(0,0) = \infty\,.$
- *Example:* $g(z) = (x,y)$ corresponds to $f(z) = \overline{z}\,.$ Then $f(z)$ is *not* analytic, and $g(x,y)$ does *not* satisfy Maxwell's equations, since $\operatorname{curl} g(x,y) = 0$ but $\operatorname{div} g(x,y) = 2$ everywhere.

3. Parametrized curves in the plane

- *Parametrized curve:* $\mathcal{C} = c(t) = (x(t), y(t))$ for $a \le t \le b$.
  We can imagine $c(t)$ as the position at time t of a particle moving along $\mathcal{C}$ from the start point $c(a) = (x(a), y(a))$ to the end point $c(b) = (x(b), y(b))\,.$ $\mathcal{C}$ is a *closed curve* if $c(a) = c(b)$.
- *Tangent vector* at point $c(t)$:

  $$c'(t) = \lim_{\epsilon \to 0} \frac{c(t+\epsilon) - c(t)}{\epsilon} = (x'(t), y'(t))\,.$$

  Rephrasing: for two points $c_0 = c(t_0)$ and $c_1 = c(t_1)$ close together along $\mathcal{C}$, the increment vector between them is approximately the velocity vector multiplied by the time increment:

  $$c_1 - c_0 \approx c'(t_1)\,(t_1 - t_0) = c'(t_1)\,\Delta t_1\,.$$

- *Example:* $\mathcal{C} = c(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$, unit circle.
  Tangent vector at $c(t)$ is: $c'(t) = (-\sin t, \cos t)\,.$
  For $t = \pi/2\,, \quad c(t) = (0,1)\,, \quad c'(t) = (-1,0)\,.$

4. Circulation around a curve

- We wish to measure the total drag or circulation of $g(x,y)$ pushing around a closed curve $\mathcal{C}$. This is a large-scale version of $\operatorname{curl} g\,,$ which measures the rate of circulation of $g(x,y)$ near a particular point.

- *Drag:* The drag of a constant vector field $g(x,y) = (c,d)$ along the line segment from $(0,0)$ to (p,q) is the dot-product:

$$(c,d) \bullet (p,q) = cp + dq\,,$$

the product of vector lengths times cos of the angle between.

- *Circulation line integral of* $g(x,y)$ along $\mathcal{C}$. Mark N points of $\mathcal{C}$:

$$c_0, c_1, \ldots, c_N = c_0\,,$$

with $c_j = c(t_j)$. We have:

$$c_j - c_{j-1} \approx c'(t_j)\,(t_j - t_{j-1}) = c'(t_j)\,\Delta t_j\,.$$

We can compute the total circulation of $g(x,y)$ around $\mathcal{C}$ by adding up the drag along each tiny line segment from c_{j-1} to c_j:

$$\begin{aligned}\oint_{\mathcal{C}} g(x,y) \bullet dc &:= \lim_{N\to\infty} \sum_{j=1}^{N} g(c_j) \bullet (c_j - c_{j-1}) \\ &:= \lim_{N\to\infty} \sum_{j=1}^{N} g(c(t_j)) \bullet c'(t_j)\ \Delta t_j \\ &= \int_{t=a}^{b} g(c(t)) \bullet c'(t)\,dt\,.\end{aligned}$$

Note that $g(c(t)) \bullet c'(t)$ is a scalar-valued function of t, so the last line is an ordinary integral.

- *Example:* Let $\mathcal{C} = (\cos t, \sin t)$ for $0 \le t \le 2\pi$, and $g(x,y) = (1,0)$ a horizontal constant vector field. Since the drag on top of the curve cancels the opposite drag on the bottom, we expect zero circulation. In fact:

$$\begin{aligned}\oint_{\mathcal{C}} g(c) \bullet dc &= \int_{t=0}^{2\pi} g(\cos t, \sin t) \bullet (\cos' t, \sin' t)\,dt \\ &= \int_{t=0}^{2\pi} (1,0) \bullet (-\sin t, \cos t)\,dt = \int_{t=0}^{2\pi} -\sin t\ dt = 0\,.\end{aligned}$$

5. Global outflow via line integrals

- We wish to measure the total outflow or flux of $g(x,y)$ across a closed curve $\mathcal{C}$. This is a large-scale version of $\operatorname{div} g(x,y)$, which measures the rate of outflow near a particular point.

- *Flux:* The flow of a constant vector field $g(x,y) = (c,d)$ across a line segment from $(0,0)$ to (p,q) is the cross-product:

$$(c,d)\times(p,q) = cq - dp\,,$$

the product of vector lengths times sin of the angle between.

- *Flux line integral of* $g(x,y)$ along $\mathcal{C}$. As before, we compute the total outflow as:

$$\oint_{\mathcal{C}} g(x,y)\times dc = \lim_{N\to\infty}\sum_{j=1}^{N} g(c_j)\times(c_j - c_{j-1}) = \int_{t=a}^{b} g(c(t))\times c'(t)\,dt\,.$$

- *Example:* Again let $\mathcal{C} = (\cos t, \sin t)$ and $g(x,y) = (1,0)$. Since inflow on the left should cancel outflow on the right, we expect zero flux. In fact:

$$\oint_{\mathcal{C}} g(c)\times dc = \int_{t=0}^{2\pi}(1,0)\times(-\sin t, \cos t)\,dt = \int_{t=0}^{2\pi}\cos t\ dt = 0\,.$$

6. Green's Theorems: global versus local

- Let R be a region on the plane whose boundary is a simple closed curve $\mathcal{C}$ (oriented counter-clockwise). Let $g(x,y)$ be vector field which is defined and differentiable at every point of R.
- *Theorem:* The circulation of g around the boundary curve is equal to the total curl of g inside the region:

$$\oint_{\mathcal{C}} g(c)\bullet dc = \iint_R \operatorname{curl} g(x,y)\ dx\,dy\,,$$

where the right side is a double integral over the region R.

- *Theorem:* The flux of g around the boundary curve is equal to the total divergence of g inside the region:

$$\oint_{\mathcal{C}} g(c)\times dc = \iint_R \operatorname{div} g(x,y)\ dx\,dy\,.$$

- *Proof:* Divide R into little regions, and write the total line integral as a sum of line integrals over tiny regions. Inside each tiny region, $g(x,y)$ can be replaced by its linear approximation, so that we can compute the tiny line integrals to be the area times $\operatorname{curl} g$ or $\operatorname{div} g$.
- *Corollary:* If $g(x,y)$ is an electical force field with $\operatorname{curl} g = \operatorname{div} g = 0$ inside the region R, then g has zero circulation and flux over the boundary curve $\mathcal{C}$:

$$\oint_{\mathcal{C}} g(c)\bullet dc = \oint_{\mathcal{C}} g(c)\times dc = 0$$