Math 418H **Fall 2005**

Lecture: Mon 10/24

1. Picturing complex functions

   - A complex function $f:\mathbb{C}\to\mathbb{C}\,,\; f(x+iy)=u(x,y)+i\,v(x,y)$ has real component $u(x,y)$ and imaginary component $v(x,y)$, where $u,v:\mathbb{R}^2\to R$ are real functions on $\mathbb{C}=\mathbb{R}^2$.
   - This is the same thing as a vector field $f:\mathbb{R}^2\to\mathbb{R}^2\,,\; f(x,y)=(u(x,y),v(x,y))$, with x-component $u(x,y)$ and y-component $v(x,y)$. This can be pictured by a field plot: draw each arrow $f(x,y)$ with its base at the point (x,y).
   - *Example 1:* The complex function $f(z)=iz$ is equivalent to the vector field: $f(x,y)=(-y,x)$ whose field plot has arrows circulating around the origin, with length proportional to their distance from the origin. This is the velocity field of a turn-table.

     For a general $\alpha=r\,\mathrm{cis}\,\theta$, the field plot of $f(z)=\alpha\,z$ is a vortex centered at the origin, with the arrows rotated by angle θ away from the outward direction, like the velocity field of water swirling down the drain.
   - *Example 2:* The complex function $f(z)=z^2+(1+i)z+1$ is equivalent to the vector field $f(x,y)=(x^2-y^2+x-y+1\,,\;2xy+y+x)$
   - *Example 3:* The complex function $f(z)=\overline{z}$, complex conjugate, is equivalent to the vector field $f(x,y)=(x,-y)$.

2. Derivative of a vector field

   - An arbitrary vector field $f(x,y)=(u(x,y),v(x,y))$ has a derivative matrix:
     $$Df:=\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix},$$
     where
     $$u_x(x,y)=\frac{\partial u}{\partial x}:=\lim_{\epsilon\to 0}\frac{u(x+\epsilon,\,y)-u(x,y)}{\epsilon}$$
     is the partial derivative of $u(x,y)$ in the x-direction, etc.
   - If $f:\mathbb{R}\to\mathbb{R}$ is an ordinary real function, its derivative $f'(a)$ gives the slope of the best linear approximation to $f(x)$ near $x=a$: for small ϵ, we have:
     $$f(a+\epsilon)\approx f(a)+f'(a)\,\epsilon\,,$$

which is just unravelling the definition of derivative:

$$f'(a) \approx \frac{f(a{+}\epsilon) - f(a)}{\epsilon}\,.$$

Similarly, for a vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$, the derivative matrix $Df(a,b)$ gives the best linear-function approximation near the point (a,b): for small (ϵ_1, ϵ_2), we have:

$$f(a{+}\epsilon_1,\, b{+}\epsilon_2) \approx f(a,b) + Df(a,b) \cdot \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix},$$

where the last operation is matrix multiplication.

- *Example 2:* For $f(x,y) = (x^2{-}y^2{+}x{-}y{+}1\,,\, 2xy{+}y{+}x)$, we have:

$$Df(x,y) = \begin{bmatrix} 2x{+}1 & 2y{+}1 \\ -2y{-}1 & 2x{+}1 \end{bmatrix}$$

- *Example 3:* For $f(x,y) = (x, -y)$, we have:

$$Df(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

3. Complex analytic functions

- We say a complex function $f(x+iy) = u(x,y) + i\,v(x,y)$ is *complex analytic* (or just *analytic*) if any of the following equivalent conditions apply.
- The partial derivatives of $f(z) = f(x+iy)$ in the real and imaginary directions are *equal*:

$$\begin{aligned} \frac{\partial f(x+iy)}{\partial x} &= \lim_{\epsilon\to 0} \frac{f(z+\epsilon) - f(z)}{\epsilon} = u_x(x,y) + iv_x(x,y) \\ \overset{!}{=} \frac{\partial f(x+iy)}{\partial iy} &= \lim_{\epsilon\to 0} \frac{f(z+i\epsilon) - f(z)}{i\epsilon} = v_y(x,y) - iu_y(x,y)\,. \end{aligned}$$

  We define the complex derivative $f'(z)$ to be the common value of these partial derivatives.
- For every value $z = x+iy$, the derivative matrix $Df(x,y)$ is a complex multiplication matrix M_{c+id} for some $c + id \in \mathbb{C}$:

$$Df := \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}.$$

We define the complex derivative $f'(z)$ to be the complex number in this multiplication matrix:

$$f'(z) := c + id = u_x + iv_x = v_y - iu_y \,.$$

- The component functions of $f(x+iy) = u(x,y) + iv(x,y)$ satisfy the *Cauchy-Riemann* partial differential equations:

$$u_x = v_y \quad , \quad v_x = -u_y \,.$$

4. Examples: analytic and non-analytic functions

- *Example 1:* $f(z) = iz\,,\;\; f(x,y) = (-y, x)\,,$

$$f'(z) = (u_x, v_x) = (v_y, -u_y) = (0,1) = i\,.$$

- *Example 2:* $f(z) = z^2{+}(1{+}1)z{+}1\,,\; f(x,y) = (x^2{-}y^2{+}x{-}y{+}1\,,\, 2xy{+}y{+}x)\,,$

$$f'(z) = (u_x, v_x) = (v_y, -u_y) = (2x+1, 2y+1) = 2z+1\,.$$

- *Example 3:* $f(z) = \overline{z}\,,\;\; f(x,y) = (x, -y)\,,$

$$f'(z) = (u_x, v_x) = (1,0) \stackrel{?}{=} (v_y, -u_y) = (-1,0)\,.$$

  The equality does *not* hold, so $f(z)$ is *not* analytic at any z!

- For a general complex analytic $f(z)$ with roots $z = r_1, \ldots, r_n$, the field plot has a vortex around each r_i which looks approximately like the vortex of $g(z) = \alpha\, z$ for $\alpha = f'(r_i)$.

5. Combining analytic functions

- $f(z) = \alpha$ (constant function) and $f(z) = z$ are analytic
- If $f(z)$ and $g(z)$ are analytic, then:
  - $f(z) + g(z)$ is analytic and $(f(z) + g(z))' = f'(z) + g'(z)$.
  - $f(z)\, g(z)$ is analytic and $(f(z)\, g(z))' = f'(z)g(z) + f(z)g'(z)$.
  - $f(z)/g(z)$ is analytic for all z where $g(z) \neq 0$, and

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}\,.$$

- *Corollary:* All polynomial functions $f(z) \in \mathbb{C}[z]$ are complex analytic for every z. All rational functions $f(z)/g(z)$ are complex analytic except at the points z where $g(z) = 0$.

6. Fundamental Theorem of Algebra

- *Theorem:* Any polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$$

of degree $n \geq 1$ has at least one complex root $z = \alpha$ with $f(\alpha) = 0$.

- This means: the field plot of any polynomial $f(z)$ has at least one vortex. The plot of a high-degree polynomial is very complicated, so this is not at all obvious!

  Alternatively: any complex polynomial of degree n can be completely split into n linear factors:

$$f(z) = a_n(z - r_1) \cdots (z - r_n)\,.$$

This will have fewer than n vortices if some of the r_i's coincide.

- *Strategy of Proof:* First, Cauchy's Mean Value Theorem says that for any circle in the complex plane, the value of an analytic function at the center is a certain average of the values on the circle.

- Next, Liouville's Theorem: Let $f(z)$ be complex analytic on the whole plane, with $\lim_{|z|\to\infty} f(z) = 0$, meaning that $f(z)$ becomes very small when z is far from the origin. Then $f(z)$ can only be the zero constant function: $f(z) = 0$ for all z.

  Proof: Consider any particular α, and take a very large circle centered at α. Given $\epsilon > 0$, by assumption we can take an α-centered circle large enough so that $|f(z)| < \epsilon$ for z on the circle. By Cauchy, the value $f(\alpha)$ is the average of the values $f(z)$ on the circle, so $|f(\alpha)| < \epsilon$. Since this is true for any $\epsilon > 0$, we must have $|f(\alpha)| = 0$, so $f(\alpha) = 0$. This holds for each $\alpha \in \mathbb{C}$.

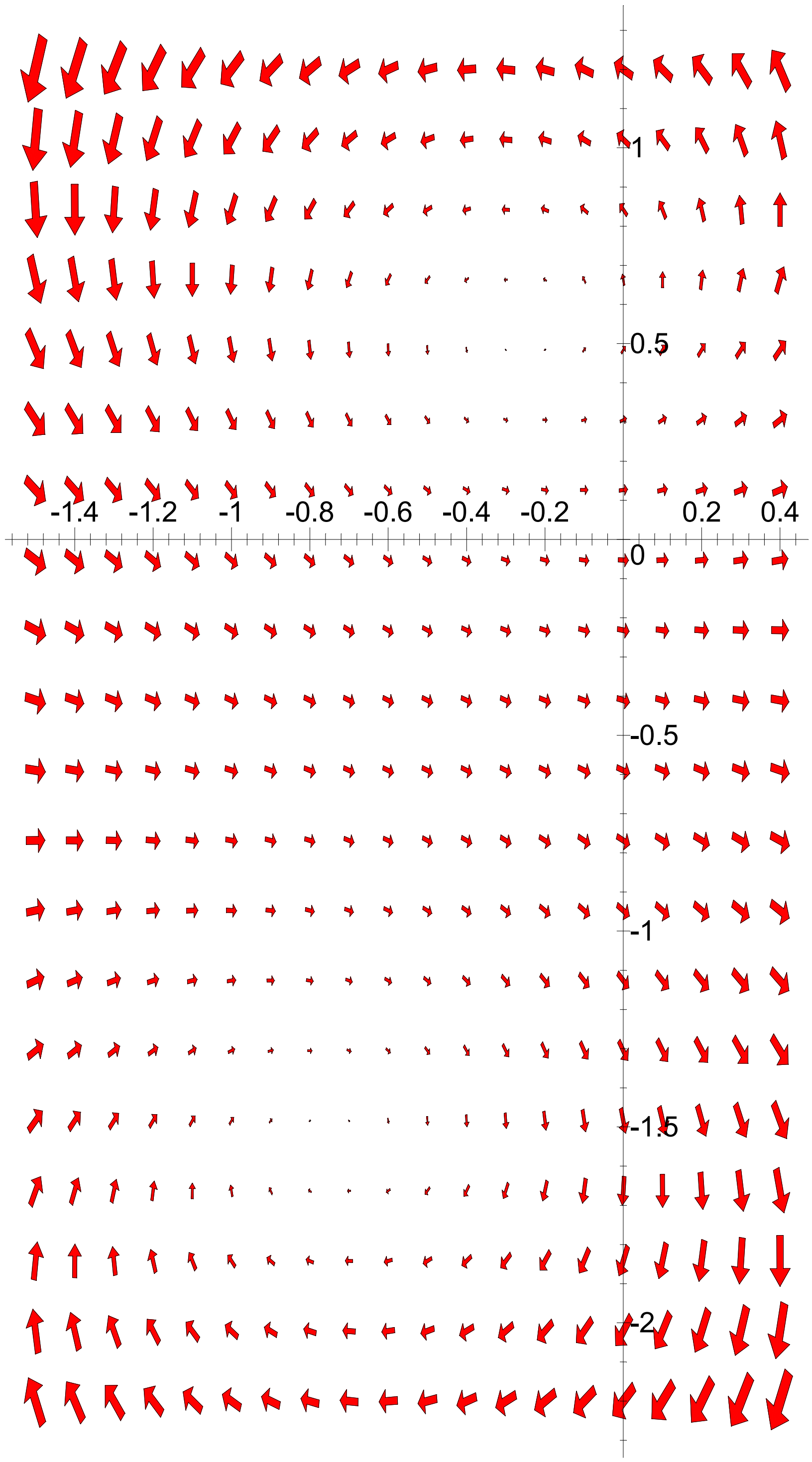
- Finally, suppose there were a polynomial function $g(z)$ with *no roots*. Then the function $f(z) = 1/g(z)$ would be analytic on the whole plane, and $|g(z)| = 1/|f(z)| \to 0$ for $|z| \to \infty$, since $\deg g(z) \geq 1$. But by Liouville, $f(z)$ can only be the zero constant function, a contradiction.

- Note that the innocent-looking *non-analytic* function:

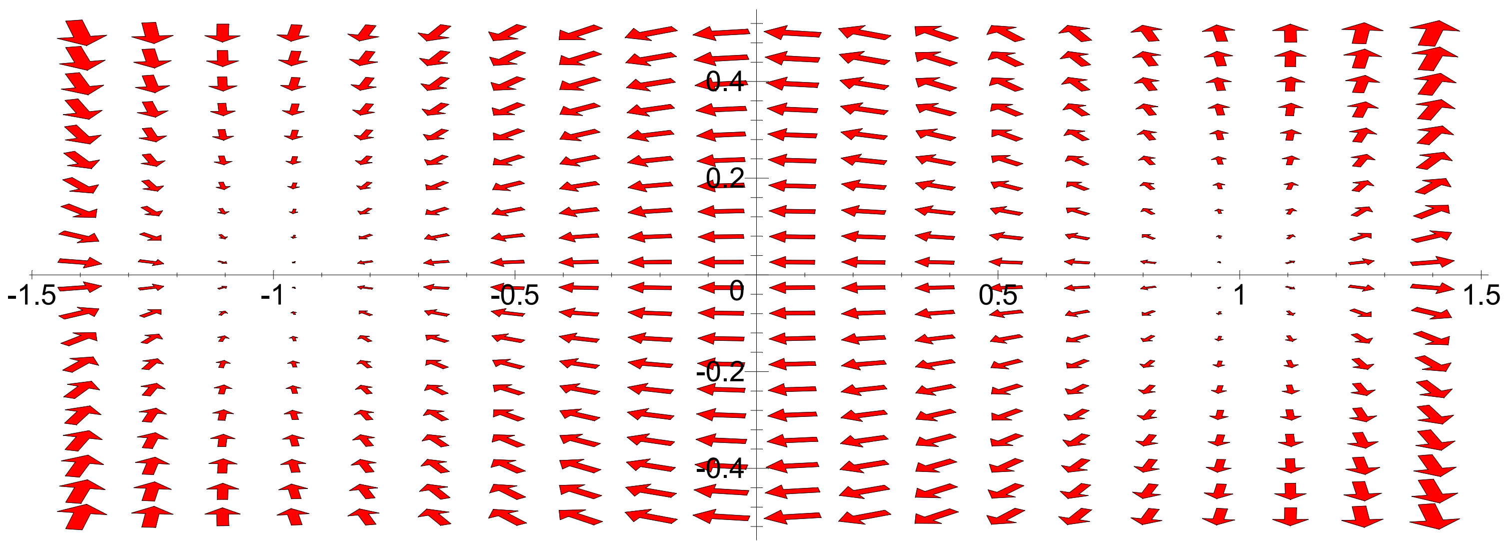
$$f(z) = z\bar{z} + 1 = |z|^2 + 1$$

has no roots! Analytic functions are very special.

$f(z) = z^2 + (1+i)z + 1$



f(z) = z^2 - 1



f(z) = z^2