Math 418H **Fall 2005**

Lecture: Wed 10/19

1. Complex multiplication = rotation

   - For $v = (a, b) \in \mathbb{C}$, consider the multiplication map

   $$\begin{array}{rcl} M_v : \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & u \bullet (x, y) \end{array}$$

   This map is $\mathbb{R}$-linear:

   $$M_v(cx, cy) = cM_v(x, y)$$

   $$M_v(x_1 + x_2, y_1 + y_2) = M_v(x_1, y_1) + M_v(x_2, y_2)\,.$$

   for all $c \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Thus:

   $$M_v(x, y) = x\, M_v(1, 0) + y\, M_v(0, 1)\,.$$

   - Multiply by $i = (0, 1)$:

   $$i \bullet (1, 0) = (0, 1) \quad , \quad i \bullet (0, 1) = (-1, 0)$$

   $$i \bullet (x, y) = \text{ rotate } (x, y) \text{ by } 90^\circ\,.$$

   - Multiply by a unit-length vector $u = cos\theta + i \sin\theta = (\cos\theta, \sin\theta)$:

   $$u \bullet (1, 0) = (\cos\theta, \sin\theta) \quad , \quad u \bullet (0, 1) = (-\sin\theta, \cos\theta)\,.$$

   $$u \bullet (x, y) = \text{ rotate } (x, y) \text{ by } \theta\,.$$

   - Write an arbitrary vector in polar coordinates: $v = ru$, where $r \in \mathbb{R}$ and $u = cos\theta + i \sin\theta$. Then:

   $$v \bullet (x, y) = \text{ rotate } (x, y) \text{ by } \theta, \text{ then stretch by } r\,.$$

2. Complex multiplication: add angles, multiply lengths

   - Consider the complex product: $v_3 = v_1 \bullet v_2$, and write each number in polar form: $v_j = r_j(\cos\theta_j + i \sin\theta_j$ for $j = 1, 2, 3$. Then:

   $$\theta_3 = \theta_1 + \theta_2 \quad , \quad r_3 = r_1 r_2\,;$$

   that is: to multiply complex numbers, add their angles and multiply their lengths.

- *First proof:* Since the multiplcation map $(x,y) \mapsto v_j \cdot (x,y)$ is rotating by θ_j and stretching by r_j, we can describe the product $v_1 \cdot v_2 = v_1 \cdot v_2 \cdot 1$ as follows: start with unit vector 1; rotate by θ_2; stretch by r_2; rotate by θ_1; stretch by r_1. Result: rotate by $\theta_1 + \theta_2$, and stretch by $r_1 r_2$.
- *Second proof:* From the formula for complex multiplication:

$$r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2\,(\,(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\,)$$

$$\stackrel{!}{=} r_1 r_2\,(\,\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\,)$$

by the angle-addition formulas.

3. Complex powers

- $2v$ is the vector v stretched by 2
- $-v$ is the vector opposite to v
- Let $v = r(\cos\theta + i\sin\theta)$.
  $v^2 = v \cdot v$ is the vector with length r^2 and angle 2θ
- $\sqrt{v}$ is a vector with length $\sqrt{r}$ and angle $\frac{1}{2}\theta$.
- There are 2 square roots because the angle θ is amiguous. We could just as well write:

$$v = r(\,\cos(\theta{+}2\pi) + i\sin(\theta{+}2\pi)\,)$$

so that

$$\begin{aligned}\sqrt{v} &= \sqrt{r}\,(\,\cos(\tfrac{1}{2}\theta{+}\pi) + i\sin(\tfrac{1}{2}\theta{+}\pi)\,)\\ &= -\sqrt{r}\,(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta)\,.\end{aligned}$$

- *DeMoivre's Theorem:* $v^{1/n}$ is any vector with length $r^{1/n}$ and angle

$$\frac{\theta + 2k\pi}{n} = \frac{\theta}{n} + \frac{2\pi k}{n}\,.$$

There are n such vectors evenly spaced around the circle, corresponding to the values $k = 0, 1, \ldots, n{-}1$.

4. Complex numbers as matrices

- Any linear mapping $M : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by a 2×2 matrix. If $M(1,0) = (a,b)$ and $M(0,1) = (c,d)$, then: $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, and:

$$M(x,y) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}.$$

  Here we use row vectors and column vectors interchangeably: $(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}$

- The linear mapping M_u for $u = \cos\theta + i \sin\theta$ is given by the matrix:

$$M_u(x,y) = v \bullet (x,y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

  This is called the *rotation matrix* of θ.

- The linear mapping M_v for $v = a + bi = ru$ is rotation by θ and stretching by r. Its matrix is:

$$M_v(x,y) = v \bullet (x,y) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

  This is called a *complex multiplication matrix*.

- Consider the set of all complex mult matrices:

$$M_{\mathbf{C}} := \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ where } a, b \in \mathbb{R} \right\}.$$

  This is a "copy" of the complex number field inside the ring of 2×2 matrices. That is, there is an isomorphism of fields from the complex numbers to this ring of matrices:

$$\begin{array}{rccc} \phi : & \mathbb{C} & \to & M_{\mathbf{C}} \\ & a + bi & \mapsto & \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \end{array}$$

  satisfies:

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2) \quad \text{and} \quad \phi(v_1 \bullet v_2) = \phi(v_1) \cdot \phi(v_2),$$

  where the operation on the left side of each equation is in $\mathbb{C}$, and the operation on the right side is an operation of matrices.