Math 418H **Fall 2005**

Lecture: Mon 10/17

1. Why bother with complex numbers $\mathbb{C}$?

   - Define new number systems to solve equations that have no solutions in old number systems.
   - $x+1=0$ has no soln in $\mathbb{N}$, so define $\mathbb{Z}$ (negative numbers)
   - $2x-1=0$ has no soln in $\mathbb{Z}$, so define $\mathbb{Q}$ (fractions)
   - x^2-2 has no soln in $\mathbb{Q}$, so define $\mathbb{R}$ (irrational numbers)
   - $x^2+1=0$ has no soln in $\mathbb{R}$, so define $\mathbb{C}$ (imaginary numbers)

2. Formal definition of $\mathbb{C}$

   - As with $\mathbb{Q}$ and $\mathbb{R}$, we do not try to uncover the "essence" of a new number like $i=\sqrt{-1}$. We just define it by enough information to determine all its properties.
   - $\mathbb{C}=\mathbb{R}\times\mathbb{R}=\{(a,b)\mid a,b\in\mathbb{R}\}$, pairs of real numbers: (a,b) represents the complex number $a+bi$.
   - Addition: $(a,b)+(c,d):=(a+c,b+d)$.

     Motivation: $(a+bi)+(c+di)=(a+c)+(b+d)i$.
   - Multiplication: $(a,b)\bullet(c,d):=(ac-bd,ad+bc)$.

     Motivation: $(a+bi)\bullet(c+di)=ac+bdi^2+adi+bci=(ac-bd)+(ad+bc)i$.
   - Check the field axioms for $\mathbb{C}$. Identity elements: $(0,0)$, $(1,0)$. Multiplicative associativity:

     $$\begin{aligned}[\,(a,b)\bullet(c,d)\,]\bullet(e,f) &= (ace-adf-bcf-bde)+(acf+ade+bce-bdf)i\\ &= (a,b)\bullet[\,(c,d)\bullet(e,f)\,]\,.\end{aligned}$$

   - The only tricky property is the existence of multiplicative inverses. We *should* have:

     $$\frac{1}{a+bi}=\frac{a-bi}{(a+bi)(a-bi)}=\frac{a-bi}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i\,.$$

     This is motivation, but proves nothing, because we have not established that $1/(a+bi)$ even exists.

- Given $(a,b) \neq (0,0)$, we *define* the multiplicative inverse as:

$$(a,b)^{-1} := \left(\frac{a}{a^2+b^2} \;,\; -\frac{b}{a^2+b^2} \right) .$$

  Now we *prove* that $(a,b) \bullet (a,b)^{-1} = (1,0)$ by applying the definition of multiplication.

- Notation: a real number $a \in \mathbb{R}$ is identified with $(a,0)$, so we can regard $\mathbb{R} \subset \mathbb{C}$. Define $i := (0,1)$.
- Prove that $i^2 = -1$ and $(a,b) = a + b \bullet i$.

3. Geometric picture of $\mathbb{C}$

- Picture: $\mathbb{C} = \mathbb{R}^2$, $x + iy = (x,y)$, vectors in the real plane
- Addition of complex numbers = usual addition of vectors (diagonal of parallelogram)
- Multiplication of complex numbers = some kind of multiplication of plane vectors:

$$(a+ib) \bullet (x+iy) = (a,b) \bullet (x,y) .$$

- Multiplying by $a = (a,0)$, we have

$$a \bullet (x,y) = (ax, ay) = \text{ stretch } (x,y) \text{ by } a ,$$

  the usual scalar multiple of a vector
- Multiplying by $i = (0,1)$, we have

$$i \bullet (1,0) = (0,1) \quad , \quad i \bullet (0,1) = (-1,0)$$

  and: $(x,y) \mapsto i \bullet (x,y) = (-y,x)$ is an $\mathbb{R}$-linear map. Thus:

$$i \bullet (x,y) = \text{ rotate } (x,y) \text{ by } 90^\circ .$$

- Multiplying by a unit-length vector $u = cos\theta + i \sin\theta = (\cos\theta, \sin\theta)$:

$$u \bullet (1,0) = (\cos\theta, \sin\theta) \quad , \quad u \bullet (0,1) = (-\sin\theta, \cos\theta)$$

  and $(x,y) \mapsto u \bullet (x,y)$ is an $\mathbb{R}$-linear map. Thus:

$$u \bullet (x,y) = \text{ rotate } (x,y) \text{ by } \theta .$$

- $$\begin{aligned}(\cos\theta + i\sin\theta)\bullet\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\end{aligned}$$

4.
-
-
-
-