Math 418H **Fall 2005**

Lecture: Wed 10/10

1. Classifying real numbers

   - $\mathbb{R} \setminus \mathbb{Q}$ are the *irrational* numbers.
   - Let A be the set of *algebraic real numbers*, those reals which are roots of some polynomial $f(x) \in \mathbb{Q}[x]$.
   - We call $\mathbb{R} \setminus A$ the *transcendental* numbers. For example, $\pi = 3.14\cdots$ is transcendental, meaning that $a_0 + a_1\pi + \cdots + a_n\pi^n \neq 0$ for any $a_0, \ldots, a_n \in \mathbb{Q}$.

2. Degrees of infinity (Georg Cantor)

   - Cardinality: Two sets are said to have the same size or cardinality if there exists a one-to-one correspondence (bijection) between them.
   - Countable: a set whose elements can be put into a list; i.e., the set has the cardinality of the natural numbers $\mathbb{N}$.
   - $\mathbb{Z}$ is countable: $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$
   - $\mathbb{Q}$ is countable: $\mathbb{Q}_{>0} = \{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \ldots\}$. In the list, skip over repeated rational numbers. Then alternate positive and negative to list all $\mathbb{Q}$.
   - A is countable by a similar argument.
   - $\mathbb{R}$ is *not* countable. Suppose we had a list $\{a_1, a_2, \ldots\}$ of *all* the real numbers in the interval $(0, 1)$. Write each number in decimal form: $a_i = 0.a_{i1}a_{i2}a_{i3}\cdots)$, where a_{ij} is a digit 0–9. Define a decimal number $b = 0.b_1b_2b_3\cdots$ by choosing the digits $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, etc. Then clearly $b \neq a_i$ for any i, since they differ in the i^{th} digit, so b is a real number *not* on the list. Therefore, there can be no such complete list.
   - The irrational numbers, and even the transcendental numbers, are uncountable, so there are much, much more of them than of rationals or algebraic numbers.

3. Uniqueness of the real numbers

   - *Theorem:* The real numbers $\mathbb{R}$ are structurally defined by the properties of a topologically complete ordered field.

     That is, if $\mathcal{R}$ is any topologically complete ordered field, then there exists a unique one-to-one correspondence $\phi : \mathbb{R} \to \mathcal{R}$ which respects addition and multiplication:

$$\phi(a + b) = \phi(a) + \phi(b) \qquad \text{and} \qquad \phi(ab) = \phi(a)\,\phi(b)\,,$$

for every $a, b \in \mathbb{R}$ (so that $\phi(a), \phi(b) \in \mathcal{R}$). We say that ϕ is an *isomorphism* of fields. Furthermore, ϕ respects order: $a < b \iff \phi(a) < \phi(b)$.

- *Proof.* First $\mathcal{R}$, being a field, has unique additive and multiplicative identity elements $\tilde{0}, \tilde{1} \in \mathcal{R}$. Now define the counterpart of an integer
$$\tilde{n} := \underbrace{1 + \cdots + 1}_{n \text{ times}} \in \mathcal{R}\,.$$
Now $\tilde{1} = \tilde{1}^2 > \tilde{0}$ in the ordered field $\mathcal{R}$, so if $n < m \in \mathbb{Z}$, then in $\mathcal{R}$:
$$\tilde{n} < \tilde{n} + \tilde{1} + \cdots + \tilde{1} = \tilde{m}\,.$$
We can now make a copy of $\mathbb{Q}$ in $\mathcal{R}$ consisting of the quantities $\tilde{n}/\tilde{m}$, and these numbers behave the same as ordinary rationals. Finally, every real number $a \in \mathbb{R}$ is the least upper bound of a cutset $S \subset \mathbb{Q}$, so define its counterpart $\tilde{a} := \text{lub}\{\tilde{s} \mid s \in S\} \in \mathcal{R}$, which exists since $\mathcal{R}$ is topologically complete. Now define $\phi : \mathbb{R} \to \mathcal{R}$ by $\phi(a) := \tilde{a}$. We may show this has the desired properties, and is unique.

4. Exercise: $\mathbb{Z}$ is topologically complete

   - We check the least upper bound property. Let $A \subset \mathbb{Z}$ be a bounded, non-empty set of integers with upper bound $r \in \mathbb{Z}$. For $a \in A$, the subset $A \cap [a, r] = \{a_1, \ldots, a_n\}$ has at most $r - a$ elements. We clearly have $m = \max(a_1, \ldots, a_n) = \max A$, and this is the least upper bound of A in $\mathbb{Z}$.

5. Exercise: If $f(x), g(x)$ are continuous functions at $x = a$, then the product function $f(x)g(x)$ is likewise.

   - We want to control the deviation $|f(x)g(x) - f(a)g(a)|$ in terms of $|f(x) - f(a)|$ and $|g(x) - g(a)|$. We have:
   $$\begin{aligned} |f(x)g(x) - f(a)g(a)| &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\ &\leq |f(x)|\,|g(x){-}g(a)| + |f(x){-}f(a)|\,|g(a)| \end{aligned}$$
   - Given $\epsilon > 0$, choose $\delta > 0$ small enough so that
   $$|f(x) - f(a)| < \min\left(\frac{\epsilon}{2(|g(a)| + \epsilon)}, \epsilon\right),$$
   $$|g(x) - g(a)| < \frac{\epsilon}{2(|f(a)| + \epsilon)}\,.$$
   Then we have $|f(x)| \leq |f(a)| + \epsilon$, and:
   $$\begin{aligned} |f(x)g(x) - f(a)g(a)| &< (|f(a)|{+}\epsilon)\,\frac{\epsilon}{2(|f(a)|{+}\epsilon)} + |g(a)|\frac{\epsilon}{2(|g(a)| + \epsilon)} \\ &< \epsilon/2 + \epsilon/2 = \epsilon\,. \end{aligned}$$