Math 418H **Fall 2005**

FINAL REVIEW HW

1a. Use the Euclidean algorithm to compute the greatest common divisor $d = \gcd(372, 264)$.

b. Find integers n, m such that $d = 372n + 264m$.

2. Prove the following assuming only the Euclidean algorithm and its immediate consequences.

a. *Lemma:* Let $a, b, c, d \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and suppose $ad = bc$. Then $a|c$ and $b|d$.

b. *Lemma:* Every rational number has a unique expression in lowest terms. That is, if we have $a/b = c/d$, with positive integers $a, b, c, d \in \mathbb{Z}$ and $\gcd(a, b) = \gcd(c, d) = 1$, then $a = c$ and $b = d$.

c. *Proposition:* If $a/b \in \mathbb{Q}$ is a fraction in lowest terms with $(a/b)^{n/m} \in \mathbb{Q}$, then $a^{n/m}, b^{n/m} \in \mathbb{Z}$.

d. Show that $\sqrt[3]{2} \,/\, \sqrt[5]{6}$ is irrational.

3. Let $f(x) = 3x^5 - 4x^4 + 2x^3 - 5x^2 - 12x - 4$.

a. Find all rational roots of $f(x)$ by means of the Rational Root Test.

b. Find all irrational real roots of $f(x)$, accurate to 1 decimal place, by applying Newton's Method over the reals. Hint: Divide out the linear factors found in part (a) before applying Newton.

c. Find all non-real complex roots of $f(x)$, accurate to 1 decimal place in each component. Hint: Divide out by the real linear factor found above, and in Newton's Method take the initial guess $x = x_0$ to be non-real.

4. Use the definition of continuity to prove that $f(x) = 1/x^2$ is continuous at $x = 3$.

5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an automorphism of the real numbers: a one-to-one and onto correspondence which respects addition and multiplication: $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\,\phi(b)$ for all $a, b \in \mathbb{R}$.

a. Show that $\phi(m/n) = m/n$ for every rational number $m/n \in \mathbb{Q}$. Hint: $\phi(1) = 1$, so

$$\phi(\underbrace{1 + \cdots + 1}_{n \text{ times}}) = \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

b. Show that if $a < b$, then $\phi(a) < \phi(b)$ for every $a, b \in \mathbb{R}$. Hint: $a < b$ iff $b - a = c^2$ for some $c \in \mathbb{R}$.

c. Show that $\phi(a) = a$ for all $a \in \mathbb{R}$. Hint: Suppose $\phi(a) < a$, and consider a rational number with $\phi(a) < m/n < a$.

6. In the complex plane, let $p_1, \ldots, p_5$ be the vertices of a regular pentagon with center at $q = 1+i$, the corner $p_1 = 0$ being at the origin. Write the polynomial $f(z) = (z - p_1) \cdots (z - p_5)$ as explicitly as possible.

7. Let $f(z) = \frac{1}{z+1}$.

a. Explicitly verify the Cauchy-Riemann equations which guarantee $f(z)$ is an analytic function except at $z = 1$.

b. In the complex plane, let $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$ be the closed curve whose three pieces are line segments from 0 to 1, from 1 to i, and from i back to 0: the sides of an isoceles right triangle. Explicitly compute the complex line integral $\oint_{\mathcal{T}} f(z)\, dz$, verifying Cauchy's vanishing theorem.

c. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the circles with center $1+i$ and radii $1/2$ and 2 respectively. Explicitly verify that the Cauchy Vanishing and Cauchy Mean Value theorems hold for $f(z)$ on $\mathcal{C}_1$, but not for $f(z)$ on $\mathcal{C}_2$. (Why not?)