MTH 418H **Fall 2005**

Lecture Mon 10/4/05

Algebra Definitions 2: Real Numbers

- There is not necessarily any natural order on a given commutative ring R: rather, we must define it. An **order relation** on R is a specification of when $a < b$ holds for elements $a, b \in R$. Once $<$ is defined, we let $a > b$ mean $b < a$, and we let $a \leq b$ mean $a < b$ or $a = b$. The defined relation must obey the following axioms:

  (i) Compatibility with $+$ and $\cdot$
  If $a < b$ and c is arbitrary, then $a + c < b + c$.
  If $0 < a < b$ and $0 < c$, then $a \cdot c < b \cdot c$.

  (ii) Trichotomy: For any $a \in R$, exactly one of the following holds: $a > 0$, $a = 0$ or $a < 0$.

  EXERCISES: These axioms imply all the usual algebraic properties of inequalities. Prove the follwing:

  - $a < b \iff b - a > 0$
  - If $a < b$ and $b < c$, then $a < c$.
  - If $a > 0$, then $-a < 0$.
  - If $a, b < 0$, then $ab > 0$.
  - If R contains an element with $a^2 = -1$, then there is no possible order relation on R. (Thus, there is no possible order on the complex numbers $R = \mathbb{C}$.)

- Consider an ordered ring R. An *upper bound* of a subset $A \subset R$ is an element $b \in R$ such that $b \geq a$ for all $a \in A$. A *least upper bound* of A is an upper bound b such that $b \leq b'$ for every upper bound b' of A.

  We say that R is **topologically complete** if it obeys the *least upper bound property*:
  If a set A has any upper bound in $r \in R$, then A has a least upper bound in $r' \in R$.

  EXERCISES:

  - The field of rational numbers $R = \mathbb{Q}$ is *not* topologically complete. Answer: The set $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ has upper bounds 1.5, 1.42, 1.415, etc., but does not have any least upper bound in $\mathbb{Q}$.
  - The ring of integers $R = \mathbb{Z}$ *is* topologically complete.

- We **construct the field of real numbers** $\mathbb{R}$ out of the rational numbers $\mathbb{Q}$ by defining a real number to be a *cutset*: i.e., a set of rational numbers $S \subset \mathbb{Q}$ such that:

  (i) S is a downset: $s \in S$ implies $t \in S$ for all $t < s$.

  (ii) S is non-trivial: $S \neq \emptyset, \mathbb{Q}$.

  (iii) S contains no maximal element: no element $s \in S$ is an upper bound of S.

  Defining $+$, $\cdot$, and $<$ appropriately, we show that $\mathbb{R}$ is a topologically complete, ordered field.

- Addition: $S+T := \{s+t \mid s\in S\,,\, t\in T\}\,.$
- Zero element: $S_0 = \mathbb{Q}_{<0} := \{s\in\mathbb{Q} \mid s<0\}$.
- Negatives: $-S := \{-s \mid s\notin S\,, s\neq \mathrm{lub}(S)\}\,.$
- Order: $S<T$ means $S\subset T$
- Multiplication: For $S,T\geq S_0$, define:

$$S\bullet T := \{st \mid s\in S\,,\, t\in T\,,\, s,t\geq 0\} \,\cup\, S_0\,.$$

  For $S<0<T$, define $S\bullet T := -(-S\bullet T)$, and similarly for other cases.

We then proceed to prove that the above definition satisfies the properties of a field with order and topological completeness. This involves a lot of checking, but our definitions at least make the completeness easy: If $\mathcal{A}\subset\mathbb{R}$ is any collection of downsets $S\in\mathcal{A}$, then an upper bound is a cutset $B\subset\mathbb{Q}$ with $S\subset B$ for all $S\in\mathcal{A}$. Then we easily check that $B:=\bigcup_{S\in\mathcal{A}} S$ is a cutset, and is the least upper bound of $\mathcal{A}$.

Our definition establishes the existence of $\mathbb{R}$, but once we have established it, we *never* use it in proofs. Rather, we rely on the *unique properties* of $\mathbb{R}$ stated in the following result.

- **Theorem** If R is any topologically complete ordered field, then R is naturally isomorphic to $\mathbb{R}$. That is, there is a unique map $\phi : R\to\mathbb{R}$ which is one-to-one and onto, and which respects addition and multiplication: $\phi(a+b)=\phi(a)+\phi(b)$ and $\phi(a\cdot b)=\phi(a)\cdot\phi(b)$ for all $a,b\in R$.

  That is, any topologically complete ordered field is just a "copy" of the real numbers, so that anything true about $\mathbb{R}$ also holds for any such field. Thus, in proving things about $\mathbb{R}$, we should only use the properties of a complete ordered field, never any specific construction of $\mathbb{R}$ such as the one above.

- A function $f:\mathbb{R}\to\mathbb{R}$ is **continuous** at $x=a$ if, for any y-tolerance $\epsilon>0$, there is some sufficiently small x-tolerance $\delta>0$ such that x being within distance δ of a guarantees that $f(x)$ is within distance ϵ of $f(a)$. That is:

$$\forall\epsilon{>}0\ \exists\delta{>}0\ :\ |x-a|<\delta \implies |f(x)-f(a)|<\epsilon\,.$$

  We have:

  - $f(x)=\text{const}$ and $f(x)=x$ are continuous at all $x=a$.
  - If $f(x), g(x)$ are continuous at $x=a$, then so are $f(x){+}g(x)$, $f(x){\cdot}g(x)$, and $f(x)/g(x)$ (the last provided $g(a)\neq 0$).
  - Any polynomial function $f(x)\in\mathbb{R}[x]$ is continuous at all $x=a$, and any rational function $f(x)/g(x)\in\mathbb{R}(x)$ is continuous at all $x=a$ with $g(a)\neq 0$.

- **Theorem** (Intermediate Value Theorem) If $f:[a,b]\to\mathbb{R}$ is a function continuous on an interval $[a,b]$, and $f(a)<v<f(b)$, then there is some value $c\in[a,b]$ such that $f(c)=v$.

  That is, $f(x)$ cannot go past the value v without hitting it. This implies that any odd-degree polynomial $f(x)\in\mathbb{R}[x]$ has a root $f(c)=0$.