**MTH 418H** **Fall 2005**

# Algebra Definitions 1

We define some terms concerning generalized number systems.

- A **ring** is a set $R$ along with operations of addition $+ : R \times R \to R$ and multiplication $\bullet : R \times R \to R$, satisfying the following properties:

  (i) + associativity: $(a+b)+c = a+(b+c)$ for all $a, b, c \in R$ .

  (ii) + identity: there exists $0 \in R$ such that $0 + a = a + 0 = a$ for all $a \in R$ .

  (iii) + inverse: for any $a \in R$, there is a $b \in R$ with $a + b = b + a = 0$ : we denote $b$ by $-a$ .

  (iv) + commutativity: $a + b = b + a$ for all $a, b \in R$ .

  (i′) $\bullet$ associativity: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all $a, b, c \in R$ .

  (ii′) $\bullet$ identity: there exists $1 \in R$ such that $1 \bullet a = a \bullet 1 = a$ for all $a \in R$ .

  (v) distributivity: $a \bullet (b + c) = a \bullet b + a \bullet c$ and $(a + b) \bullet c = a \bullet c + b \bullet c$ .

- A **division ring** is a ring satisfying:

  (iii′) $\bullet$ inverse: for any non-zero $a \in R$ , there is a $b \in R$ with $a \bullet b = b \bullet a = 0$ : we denote $b$ by $a^{-1}$ or $1/a$ .

- A **commutative ring** is a ring satisfying:

  (iv′) $\bullet$ commutativity: $a \bullet b = b \bullet a$ for all $a, b \in R$ .

- A **field** is a ring satisfying both (iii′) and (iv′).

- A **unit** in ring $R$ is an element $a$ which has a mulitiplicative inverse $a^{-1} \in R$ . Thus, a field is a ring in which every non-zero element is a unit.

- A **zero-divisor** in a ring $R$ is an element $a \neq 0$ such that $a \bullet b = 0$ for some $b \in R$ . A **domain** is a commutative ring with no zero-divisors.

- A **Euclidean ring** is a domain $R$ along with a function

  $$\text{size} : R \setminus \{0\} \to \mathbb{N}$$

  (where $\mathbb{N} = \{0, 1, 2, \cdots\}$) such that for any $a, b \in R$, there are $q, r \in R$ with $a = qb + r$ and $r = 0$ or $\text{size}(r) < \text{size}(b)$.