Normal Subgroups and Ideals

The idea of normal subgroup in a group and ideal in a ring are clearly analogous, each being the kernel of the respective kind of homomorphism. We give a more direct connection between the normal subgroups of a finite group and certain special ideals of its group ring.

Group ring and normal subgroups. Let $G$ be a finite group with identity element 1. The group ring is $R = \mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}g$, formal linear combinations of group elements, linearly extending the group multiplication. For any subset $S \subset G$, define its average element:

$$e_S = \frac{1}{|S|} \sum_{s \in S} s.$$

For a subgroup $H \subset G$, the average element is idempotent since:

$$e_H^2 = \frac{1}{|H|^2} \sum_{h' h'' \in H} h' h'' = \frac{1}{|H|^2} \sum_{h \in H} c_h h = e_H,$$

where $c_h = \# \{(h', h'') \mid h' h'' = h\} = |H|$. The $G$-representation $\mathbb{C}[G/H] = \text{Ind}_G^H(\mathbb{C}_1)$, induced from the trivial irreducible $\mathbb{C}_1$, is isomorphic to the multiplication representation on the left ideal $\mathbb{C}[G] e_H$.

If $H = N$ is a normal subgroup, then $gN = Ng$ and $g e_N = e_N g$ for all $g \in G$, so $e_N$ is a central idempotent. We thus obtain a two-sided ideal isomorphic to the group ring of the quotient group $G/N$:

$$(e_N) = \mathbb{C}[G] e_N = e_N \mathbb{C}[G] \cong \mathbb{C}[G/N],$$

with $\mathbb{C}$-basis $\{g_i e_N = e_{[g_i]}\}$ where $[g_i] = g_i N$ runs over distinct cosets in $G/N$.

Multiplication by $e_N$ projects any $N$-representation onto its component of $N$-invariant vectors, and indeed the natural projection:

$$\pi : \mathbb{C}[G] \to \mathbb{C}[G/N], \quad g \mapsto [g]$$

is isomorphic to $g \mapsto g e_N$, and its kernel is generated by the complementary central idempotent $e_N^+ = 1 - e_N$, or equivalently by

$$e_N - 1 = \frac{1}{|N|} \sum_{1 \neq n \in N} n - 1,$$

so that:

$$\mathbb{C}[G/N] \cong \frac{\mathbb{C}[G]}{\mathbb{C}[G](e_N - 1)}.$$

The vector space dimension of $\text{Ker}(\pi)$ is:

$$|G| - \frac{|G|}{|N|} = \frac{|G|}{|N|} (|N| - 1),$$

and a basis is given by:

$$\{g_i (n - 1) \mid [g_i] \in G/N \text{ and } 1 \neq n \in N\}.$$
Lie analogy. Recall that for a Lie group $G$, each connected normal subgroup $N \subset G$ corresponds to a Lie ideal $n = \text{Lie}(N)$ in the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. This generates a ring ideal in the universal enveloping algebra, and

$$U(\mathfrak{g}/n) \cong U(\mathfrak{g})/U(\mathfrak{g})n.$$  

In our case, we may consider the set $N-1 = \{n-1\}_{n \in N}$ as vaguely analogous to a Lie algebra for the group $N$, since it is mapped to $N$ by the truncated exponential $\exp_1(x) = 1 + x$, and $\mathbb{C}[G/N] \cong \mathbb{C}[G]/\mathbb{C}[G](N-1)$.

Irreducible components. The group $G$ has irreducible complex linear representations $V_\lambda$, where $\lambda$ runs over an index set of size equal to the number of conjugacy classes in $G$. We have the irreducible character functions $\chi_\lambda(g) = \text{trace}(g \mid V_\lambda)$, which we can consider as elements of $\mathbb{C}[G]$:

$$\chi_\lambda = \sum_{g \in G} \chi_\lambda(g) g.$$

Brauer’s Decomposition Theorem tells us that $R = \mathbb{C}[G]$ is a semi-simple ring with one matrix algebra component $M_{d_\lambda}(\mathbb{C})$ for each irreducible $\lambda$. Specifically, we have the irreducible central idempotents

$$e_\lambda = \frac{d_\lambda}{|G|} \chi_\lambda$$

generating the two-sided ideals

$$(e_\lambda) = Re_\lambda = e_\lambda R \cong M_{d_\lambda}(\mathbb{C}),$$

with $1 = \sum_\lambda e_\lambda$ and $R = \bigoplus_\lambda (e_\lambda)$.

The irreducible representations of $G/N$ are precisely the $G$-representations $V_\lambda$ which are fixed by $N$, i.e. Res$^G_N(V_\lambda) \cong \mathbb{C}^{d_\lambda}_1$ where $\mathbb{C}_1$ is the trivial irreducible. Thus the group ring $\mathbb{C}[G/N]$, identified with the ideal $(e_N) \subset \mathbb{C}[G]$, is the sum of the Brauer components $(e_\lambda)$ of all $N$-fixed $V_\lambda$:

$$\mathbb{C}[G/N] \cong (e_N) = \bigoplus_{N\text{-fixed } \lambda} (e_\lambda),$$

and

$$e_N = \sum_{N\text{-fixed } \lambda} e_\lambda.$$  

Correspondingly, $\text{Ker}(\pi) = (e_N-1) = \bigoplus_\mu (e_\mu)$ and $e_N-1 = \sum_\mu e_\mu$, where $\mu$ runs over irreducible $G$-representations which are $N$-moved (not $N$-fixed).

Summary. A normal subgroup $N \subset G$ corresponds to a decomposition into two-sided ideals:

$$\mathbb{C}[G] = \mathbb{C}[G](e_N-1) \oplus \mathbb{C}[G]e_N,$$

for the central idempotents

$$e_N = \frac{1}{|N|} \sum_{n \in N} n = \sum_{N\text{-fixed } \lambda} \frac{d_\lambda}{|G|} \chi_\lambda, \quad e_N-1 = \frac{1}{|N|} \sum_{n \in N} (n-1) = \sum_{N\text{-moved } \mu} \frac{d_\mu}{|G|} \chi_\mu.$$

The components are:

$$\mathbb{C}[G](e_N-1) = \text{Span}_\mathbb{C}\{g_i(n-1) \text{ for } [g_i] \in G/N, 1 \neq n \in N\},$$

$$\mathbb{C}[G/N] \cong \mathbb{C}[G]e_N \cong \frac{\mathbb{C}[G]}{\mathbb{C}[G](e_N-1)} \cong \frac{\mathbb{C}[G]}{\mathbb{C}[G](N-1)}.$$
Example. Consider the smallest non-abelian group:

$$G = S_3 = D_3 = \{1, r, r^2, s, sr, sr^2\}$$

with $r^3 = s^2 = 1$ and $rs = sr^{-1}$. The character table is:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$r, r^2$</th>
<th>$s, sr, sr^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_+$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_-$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

The irreducible central idempotents are:

$$e_+ = \frac{1}{6}(1 + r + r^2 + s + sr + sr^2)$$
$$e_- = \frac{1}{6}(1 + r + r^2 - s - sr - sr^2)$$
$$e_2 = \frac{1}{3}(2 - r - r^2),$$

The normal subgroups correspond to the following quotients of $\mathbb{C}[G]$:

- $N = \{1\}$ has $e_N = 1 = e_+ + e_- + e_2$ with $\mathbb{C}[G/N] = \mathbb{C}[G]$. Here all irreducibles are $N$-fixed.

- $N = G$ has $e_N = e_+$, since only the trivial irreducible is $N$-fixed. Also:
  $$e_N - 1 = \frac{1}{6}((r-1)+(r^2-1)+(s-1)+(sr-1)+(sr^2-1)) = - (e_+ + e_2),$$
  $$\mathbb{C}[G/N] = \mathbb{C}e_+, \quad \mathbb{C}[G](e_N - 1) = \text{Span}_\mathbb{C}\{r-1, r^2-1, s-1, sr-1, sr^2-1\}$$
  $$\mathbb{C}[G/N] \cong \frac{\mathbb{C}[G]}{(r^2 + r + s + sr + sr^2 - 5)}.$$ 

- $N = \{1, r, r^2\}$ has: $e_N = \frac{1}{3}(1 + r + r^2) = e_+ + e_-$, since $V_+, V_-$ are the $N$-fixed irreducibles. Also:
  $$e_N - 1 = \frac{1}{3}(r-1+r^2-1) = -e_2,$$
  $$\mathbb{C}[G/N] \cong \mathbb{C}[G](e_N) = \text{Span}_\mathbb{C}\{e_N, se_N\},$$
  $$\mathbb{C}[G](e_N - 1) = \text{Span}_\mathbb{C}\{r-1, r^2-1, s(r-1), s(r^2-1)\}.$$ 
  Thus:
  $$\mathbb{C}[G/N] \cong \frac{\mathbb{C}[G]}{(r + r^2 - 2)} = \frac{\mathbb{C}[G]}{(r-1, r^2-1)}$$
**Scratchwork.** We have the $G$-invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}[G]$:

$$\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

The group has irreducible complex linear representations $V_\lambda$ of dimension $d_\lambda$, where $\lambda$ runs over an index set $\hat{G}$. The number of irreducibles $|\hat{G}|$ is equal to the number of conjugacy classes of $G$. We have the irreducible character functions $\chi_\lambda(g) = \text{trace}(g \mid V_\lambda)$, which we can consider in $\mathbb{C}[G]$. The center $Z(\mathbb{C}[G])$ has a natural orthogonal basis given by $\{e_K\}$ where $K$ runs over the conjugacy classes, and an orthonormal basis given by characters $\{\chi_\lambda\}$.

The character of the permutation representation $\mathbb{C}[G/N]$ is:

$$\chi_{G/N}(g) = \# \{ [g_i] \in G/N \text{ with } [gg_i] = [g_i] \}$$

$$= \# \{ [g_i] \in G/N \text{ with } g \in g_iNg_i^{-1} = N \}$$

$$= \begin{cases} \frac{|G|}{|N|} & \text{if } g \in N \\ 0 & \text{else.} \end{cases}$$

Thus $e_N = \frac{1}{|G|} \chi_{G/N}$, $e_\lambda = \frac{d_\lambda}{|G|} \chi_\lambda$, and $e_N = \sum_\lambda a_\lambda e_\lambda$ where:

$$a_\lambda = \frac{\chi_\lambda(\chi_{G/N})}{d_\lambda}$$

$$= \frac{1}{d_\lambda} [V_\lambda : \text{Ind}^G_N(\mathbb{C}_1)]$$

$$= \frac{1}{d_\lambda} [\text{Res}^G_N(V_\lambda) : \mathbb{C}_1]$$

$$= \frac{1}{d_\lambda |N|} \sum_{n \in N} \text{trace}(n \mid V_\lambda)$$

$$= \frac{1}{d_\lambda} \text{trace}(e_N \mid V_\lambda)$$

$$= \begin{cases} 1 & \text{if } C_1 \subset \text{Res}^G_N(V_\lambda) \\ 0 & \text{else.} \end{cases}$$

The orthogonal idempotent generating Ker($\pi$) is $e_N^\perp = 1 - e_N = \sum_\lambda b_\lambda e_\lambda$ where:

$$b_\lambda = \begin{cases} 1 & \text{if } C_1 \not\subset \text{Res}^G_N(V_\lambda) \\ 0 & \text{else.} \end{cases}$$