Math 411 9/9/2020

Review: Object $X \subset \mathbb{R}^2$
shape plane

$$\underbrace{Sym(X)}_{\text{group of symmetries}} = \left\{ L : X \to X \right\}$$

invertible
linear
orthogonal = rigid, preserves length & angle

ways to move X to itself

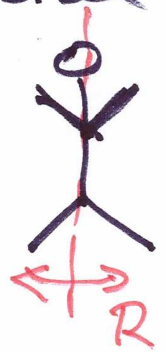
Simplest: Twofold sym

$$Sym(X) = \{ I, R \}$$

ident, trivial; extra non-trivial symmetry

Compose sym: $R \circ R = I$ (new sym) $R^2 = I$

Ex: Bilateral sym

$X =$ 

$$R = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}$$

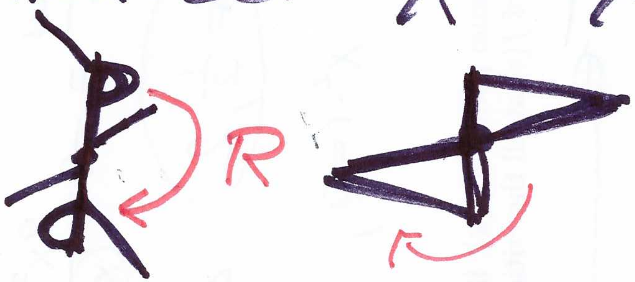
$$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

<u>Ex:</u> Point-symmetry $R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$, $R^2 = I$

Find X with $\mathrm{Sym}(X) = \{I, R\}$.

Start with asymmetric X_0,

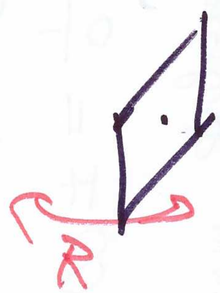
symmetrize: $X = X_0 \cup R(X_0)$



$R = \cancel{\begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix}}$
$\Downarrow$
I

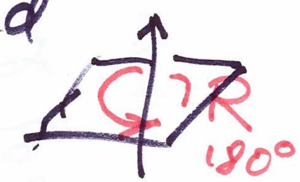
<u>Ex:</u> In $\mathbb{R}^3$, Two-fold sym.

$R = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ Mirror sym (bilateral)



$R = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$ 180° turn around axis

axis



$R = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} = -I$ Point-symmetry

surprise!

$R(\vec{v}) = -\vec{v}$

⊙ Group theory = theory of symmetry

abstract groups G

How many symmetries

composition structure

Ex: $G \to \{I, R\}$ twofold

$R \circ R = I$

representations & G

Finding objects X

with the given

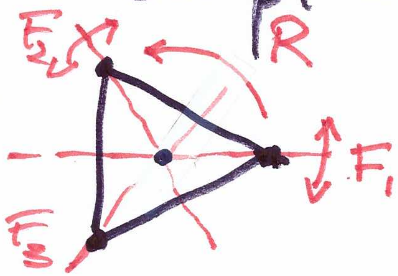
abstract sym

$G = Sym(X)$

$R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ 

$R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ 

⊙ More complicated symmetry.

$X =$ 

$$G = Sym(X) = \left\{ \begin{matrix} I, & R, & R^2 \\ F_1, & F_2, & F_3 \\ A & B & C \end{matrix} \right\}$$

$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Dihedral gp. D_3
of triangle

$R^2 = R \circ R = 240°$ rot

$R^3 = I$

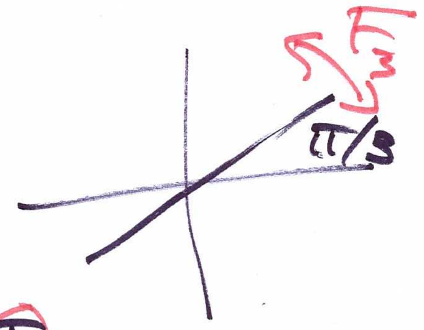
$R = Rot_{\frac{2\pi}{3}}$

$R^3(\vec{v}) = R(R(R(\vec{v}))) = \vec{v}$

$I(\vec{v}) = \vec{v}$

$F_1^2 = I$, $\quad R = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}$

$F_1 \cdot R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$



$A \cdot R = C$

$$= \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} = Fl_{\frac{\pi}{-3}} = F_2$$

$F_1 R = F_3$

Mult table

Rotations only

	I	R	R^2	A	B	C
I	I	R	R^2	A	B	C
R	R	R^2	I			
R^2	R^2	I	R			
A	A	C		I		
B	B				I	
C	C					I

$I \cdot L = L$

$L \cdot I = L$

$R \cdot R^2 = R^3 = I$

$R^2 \cdot R^2 = R \cdot R^3 = R \cdot I = R$

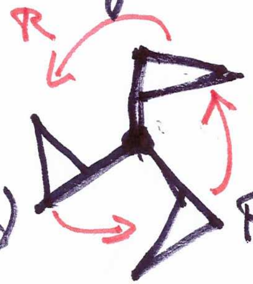
Inside $G = D_3$ have smaller group, rotations

subset $H = \{I, R, R^2\}$ closed under composition

"H subgroup of G"

$H = \mathrm{Sym}(Y)$ some object $Y \subset \mathbb{R}^2$

$Y =$



$Y = X_0 \cup R(X_0) \cup R^2(X_0)$

$R(Y) = R(X_0) \cup R^2(X_0) \cup \underbrace{R^3(X_0)}_{X_0}$