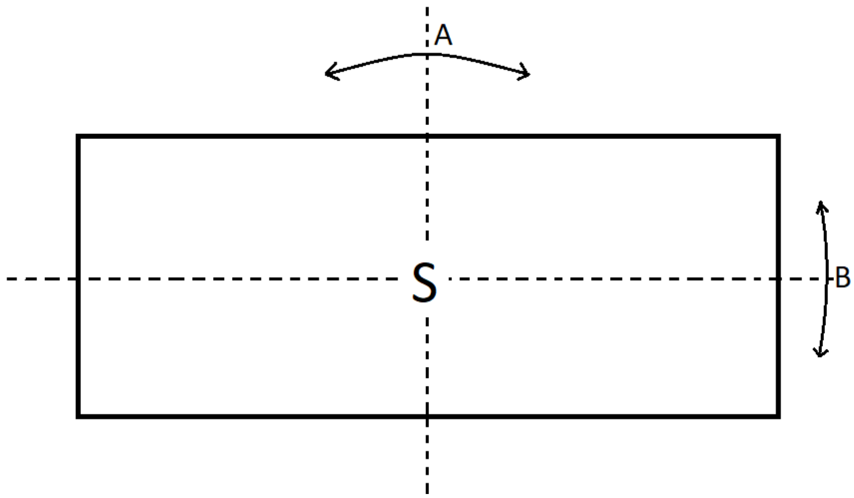
Symmetries. A *symmetry* of an object S is an invertible mapping from S to itself which preserves the structure of S. We consider the example where S is a rectangle in the plane, with the geometric structure of distance between points. The symmetries are the rigid (distance-preserving) mappings from S to itself, which are therefore linear mappings. Here are two examples:



The horizontal reflection A and the vertical reflection B both take S to itself; by contrast, a 90° rotation takes S to a different rectangle, and is *not* a symmetry. Taking the center of S as the origin, we can determine the matrix of A by recording the outputs $A(1,0) = (-1,0)$ and $A(0,1) = (0,1)$, so $A = \left[\begin{array}{r|r} -1 & 0 \\ 0 & 1 \end{array}\right]$; and similarly $B = \left[\begin{array}{r|r} 1 & 0 \\ 0 & -1 \end{array}\right]$.

Are these the only symmetries? We also have inverse symmetries, but $A = A^{-1}$ and $A \cdot A = I$: a reflection undoes itself, and doing it twice is the same as doing nothing. (Here I is the identity mapping with $I(v) = v$, having matrix $\left[\begin{array}{r|r} 1 & 0 \\ 0 & 1 \end{array}\right]$.) However, we can find a new symmetry by composing the two known ones, doing the horizontal reflection after the vertical one:

$$C \;=\; A \cdot B \;=\; \left[\begin{array}{r|r} -1 & 0 \\ 0 & -1 \end{array}\right].$$

What is the composite motion? Its matrix tells us that $C(1,0) = (-1,0)$ and $C(0,1) = (0,-1)$, so in fact $C(v) = -v$, and C takes each vector to its opposite. Another way to describe this is as a 180° rotation. Now we have all the symmetries.

Symmetry group. This means the set $G = \mathrm{Sym}(S)$ of all symmetries of S, endowed with the operation of *composition*, doing one symmetry after another to obtain a new symmetry. This allows us to think of G as a kind of number system, but only with multiplication, no addition. The rectangle symmetry group above is $G = \{I, A, B, C\}$ with multiplication table:

$\cdot$	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	I	C	A
C	C	B	A	I

Definition: A *group* is a set G with an operation $\cdot$ which satisfies the following axioms:

- *Closed:* For $A, B \in G$ we have $A \cdot B \in G$.
- *Associative:* $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
- *Identity:* For $A \in G$ we have $A \cdot I = I \cdot A = A$.
- *Inverses:* For $A \in G$ there is $A^{-1} = B \in G$ with $A \cdot B = B \cdot A = I$.

(These are the same axioms of multiplication as for units in a ring.) Any symmetry group is a group satisfying the four axioms, and it turns out that any abstract group satisfying these axioms is some kind of symmetry group. Note that a multiplicative group G does not have a zero element or any addition operation. Although the above example satisfies the commutative law $A \cdot B = B \cdot A$, most groups are non-commutative; a commutative group is called *abelian* (after Niels Abel).

Groups can arise from geometric symmetries as above, or from other types of symmetries. In our example, G consists of rigid motions, so it is a subgroup of the *orthogonal group* of all 2×2 orthogonal matrices,

$$\mathrm{O}(2) = \{R \in M_{2\times 2}(\mathbb{R}) \text{ with } R^T R = I\};$$

this is the rigid symmetry group of the whole plane (with the origin fixed), or of a circle centered at the origin.

G is also a subgroup of the the *general linear group* of all invertible matrices with real-number entries,

$$\mathrm{GL}_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \text{ with } A \text{ invertible}\}.$$

Like every group, $\mathrm{GL}_2(\mathbb{R})$ must be the symmetries of something. In fact, it is the symmetries of the vector space $\mathbb{R}^2$, containing all invertible mappings which preserve (respect) the vector space structure of addition and scalar multiplication (but not the geometric structure of distance and dot product).

Permutations. The most basic type of symmetry is that of an unstructured set $S = \{1, 2, \ldots, n\}$. Then $G = \mathrm{Sym}(S)$ consists of *all* invertible mappings $A : S \to S$, without any structure restrictions (not rigid, not linear, etc.). A convenient way to write such a mapping is the two-line notation with the inputs

i on the first line and the outputs $A(i)$ just below them: $A = \begin{pmatrix} 1 & 2 & \cdots & n \\ A(1) & A(2) & \cdots & A(n) \end{pmatrix}$. For example:

$$I = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,3\,4 \end{pmatrix}, \qquad A = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,1\,4\,3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,3\,2\,1 \end{pmatrix}$$

means I is the idenity with $I(i) = i$ for all i; and $A(1) = 2$, $A(2) = 1$, $A(3) = 4$, $A(4) = 3$. We can compose these to get a new permutation $C = A \circ B$ with $C(i) = A(B(i))$.

$$C = A \circ B = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,1\,4\,3 \end{pmatrix} \circ \begin{pmatrix} 1\,2\,3\,4 \\ 4\,3\,2\,1 \end{pmatrix} = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,4\,1\,2 \end{pmatrix},$$

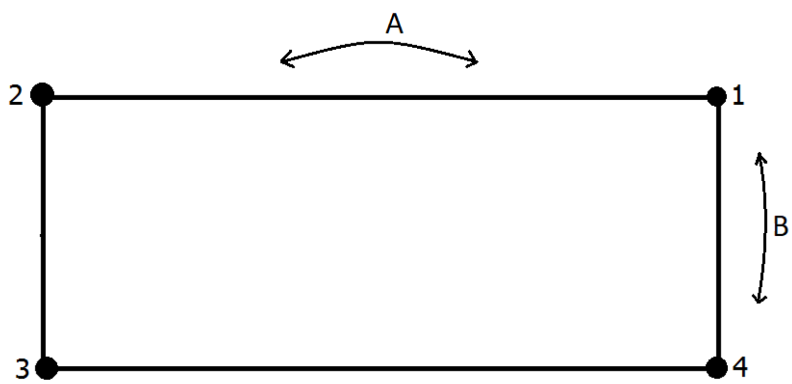
since $C(1) = A(B(1)) = A(4) = 3$ and $C(2) = A(B(2)) = A(3) = 4$, etc.

The group of all permutations of n objects, together with the composition operation, is called the *symmetric group* $G = S_n$. To construct a random permutation A, we have n choices for $A(1)$, then $n-1$ choices remaining for $A(2) \neq A(1)$, then $n-2$ choices for $A(3)$, etc., so the total number of permutations is:

$$|S_n| = n(n-1)(n-2)\cdots 1 = n!$$

This is the main significance of the factorial function.

We can realize the symmetry group of our rectangle by labeling vertices:



Then any symmetry moves the 4 vertices to each other, and the motion of the whole rectangle is determined by this permutation of $\{1, 2, 3, 4\}$. Thus, the symmetries of the rectangle described previously in geometric terms can also be written as the permutations I, A, B, C above. In fact, our symmetry group becomes a subgroup $G \subset S_4$ and we can work out its multiplication table in terms of permutations more easily than in terms of matrices.

The permutation realization of $G \subset S_4$ allows us to prove that the rectangle has no other rigid symmetries than the 4 which we have found. Indeed, imagine a symmetry T, taking vertex 1 to any vertex $T(1) = 1, 2, 3$ or 4. Then T must take the long edge $1 - 2$ to another long edge, so $T(2)$must be the only long-edge neighbor of $T(1)$. Similarly, $T(3)$ must be the only short-edge neighbor or $T(2)$, and $T(4)$ must be the only remaining vertex. That is, the 4 choices for $T(1)$ are the only choices made in constructing T, so there are at most 4 symmetries, and we have listed them all.