Math 411.001 **HW: Groups of Order 21** **Due Mon 4/29**

In the following, G denotes a group of order $|G| = 21$. Argue *briefly.*

1. What do Sylow Theorems I–III say about G, for each prime p dividing 21?

2. Show that $G = \langle x, y\rangle$ for generators x of order 3 and y of order 7.
Hint: Use the previous problem to show x and y exist, then show $x^i y^j$ for $i = 0, 1, 2$, $j = 0, 1, \ldots, 6$ make 21 *distinct* elements: $x^i y^j = x^k y^\ell$ only for $i = k$ and $j = \ell$.

3. Show that the cyclic subgroup $\langle y\rangle$ is a normal subgroup, and that $xyx^{-1} = y^j$ for $j = 1, 2$ or 4. *Hint:* What happens if you conjugate three times: $x^3yx^{-3} = \cdots$?

4. Show that if $j = 1$, then $G \cong C_3 \times C_7$, where $C_n = \{e, r, \ldots r^{n-1}\}$ with $r^n = e$.

5. Give an isomorphism mapping $C_3 \times C_7 \to C_{21}$. Indeed, we have found the *only* possible abelian group of order 21, up to isomorphism.

6. Consider the case $G = \langle x, y\rangle$ with $j = 2$, so $xy = y^2x$; and the case $\tilde{G} = \langle \tilde{x}, \tilde{y}\rangle$ with $j = 4$, so $\tilde{x}\tilde{y} = \tilde{y}^4\tilde{x}$. Give an isomorphism $\phi : G \to \tilde{G}$ by finding appropriate $x' = \phi(x)$ and $y' = \phi(y)$, and checking that $x', y' \in \tilde{G}$ obey the relations for G. We have found the *only* non-abelian group of order 21, up to isomorphism.

7. Draw the Cayley graph of the above *abelian* G, with 21 vertices $g = y^j x^i$ and colored arrows $g \xrightarrow{x} gx$ and $g \xrightarrow{y} gy$. *Hint:* Draw 3 concentric 7-cycles of y-arrows.

8. From now on, let $G = \langle x, y\rangle$ denote the *non-abelian* group with $x^3 = y^7 = e$ and $xy = y^2x$. Draw the Cayley graph of G. *Hint:* Start with the same 3 concentric rings of 7 elements, but now the y-arrows skip around each ring.

9. Give the 8×8 multiplication table of the finite field $\mathbb{F}_8 = \mathbb{Z}_2[\alpha]$, where α satisfies the minimal polynomial $\alpha^3 + \alpha + 1 = 0$. Remember $2 = 0$ and $-1 = 1$.

10. Recall the Frobenius mapping $\chi : \mathbb{F}_8 \to \mathbb{F}_8$ defined by $\chi(\gamma) = \gamma^2$; it is a field isomorphism because $(a+b)^2 = a^2 + b^2$. (χ is pronounced *chi.*) If we consider $\mathbb{F}_8$ as a 3-dimensional vector space with basis $\{1, \alpha, \alpha^2\}$ over the base field $\mathbb{Z}_2$, then χ becomes a linear mapping. *Problem:* Give its 3×3 matrix with entries in $\mathbb{Z}_2$.

11. Define the mapping $\psi : \mathbb{F}_8 \to \mathbb{F}_8$ by $\psi(\gamma) = \alpha\gamma$. Show this is a linear mapping over the base field $\mathbb{Z}_2$, and give its 3×3 matrix. (ψ is is pronounced *psi.*)

12. Show $\chi^3 = I$, where I is the identity mapping: doing χ three times takes each $\gamma \in \mathbb{F}_8$ back to itself; and similarly $\psi^7 = I$ and $\chi\psi = \psi^2\chi$. (Compute using the definitions of χ, ψ rather than the matrices.) Find a group isomorphic to G, consisting of 21 3×3 matrices with $\mathbb{Z}_2$ entries. (Mulitply matrices with Wolfram.)

13. The mappings χ, ψ permute the 7 non-zero elements of $\mathbb{F}_8$. Number these elements, and write χ and ψ as permutations in S_7, using cycle notation. Then how does G relate to S_7?