Math 309.005 **HW 1/19 Solution** **Spring 2011**

PROPOSITION: The only pure magic square of order 3 is the following, together with its flips and turns:

2	7	6
9	5	1
4	3	8

Proof: We must classify all 3×3 arrays of numbers:

a	b	c
d	e	f
g	h	i

such that each number $1, 2, \ldots, 9$ appears once in the array, and each row, column, and main diagonal sums to the same number n:

$$\begin{aligned} a+b+c&=n \\ d+e+f&=n \\ g+h+i&=n \end{aligned} \qquad \begin{aligned} a+d+g&=n \\ b+e+h&=n \\ c+f+i&=n \end{aligned} \qquad \begin{aligned} a+e+i&=n \\ c+e+g&=n \end{aligned}$$

Since $3n = a+b+c+\cdots+i = 1+2+\cdots+9 = 45$, we have $n = 15$.

Now consider all the ways to obtain 15 as a sum of three distinct numbers from 1 to 9. These triples are easily listed by specifying the largest number, then writing the other two in decreasing order:

$$9{+}5{+}1, \quad 9{+}4{+}2, \quad 8{+}6{+}1, \quad 8{+}5{+}2, \quad 8{+}4{+}3, \quad 7{+}6{+}2, \quad 7{+}5{+}3, \quad 6{+}5{+}4.$$

These 8 triples must correspond somehow to the 8 equations $a+b+c=15$, etc. For each number 1,. . . ,9, we ask: how many triples is it involved in? And for each letter $a, \ldots, i$ we ask: how many equations is it involved in? The results are:

four triples/eqns	5	e
three triples/eqns	$2, 4, 6, 8$	a, c, g, i
two triples/eqns	$1, 3, 7, 9$	b, d, f, i

Since e is involved in four equations (with distinct letters representing distinct numbers), its value must be a number involved in four distinct triples, meaning $e = 5$.

Similarly, each of the letters a, c, g, i represents a number involved in *at least* three triples. Since $e = 5$ is already taken, we can only have $\{a, c, g, i\} = \{2, 4, 6, 8\}$. Since $e = 5$ is in the middle, 2 must be diagonal to 8, and 4 must be diagonal to 6. A turn will bring 2 to the upper right corner, and a possible flip will bring 6 to the upper left:

2	b	6
d	5	f
4	h	8

But now we can solve for the remaining letters, obtaining the square in the Proposition.