Math 299 **Recitation 7** **Oct. 17, 2013**

Solutions

PROBLEM 1. Prove that if x is a positive real number, then $x + \frac{1}{x} \geq 2$. (hint: Experiment starting from the inequality you want to prove and derive a true statement; then try to work backwards, reversing your argument).

Proof. Since the square of any real number is nonnegative, we have

$$\begin{aligned} (x-1)^2 &\geq 0 \\ x^2 - 2x + 1 &\geq 0 \\ x^2 + 1 &\geq 2x \\ x + \frac{1}{x} &\geq 2 \qquad \text{(dividing both sides by the positive real number } x\text{).} \end{aligned}$$

□

PROBLEM 2.

1. To prove $A \implies B$ by contradiction, what do you assume (for the purpose of deriving a contradiction)?

   Answer. One assumes: A and not(B). □

2. Prove that if $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$. (hint: in trying to derive a contradiction, first show that a is even).

   Proof. Suppose that there exists $a, b \in \mathbb{Z}$ such that $a^2 - 4b = 2$. Then $a^2 = 4b + 2 = 2(2b + 1)$. So a^2 is even. This implies that a is also even. So $a = 2k$ for some integer k. Substituting, we find that $(2k)^2 - 4b = 4k^2 - 4b = 2$. Dividing everything by 2 gives $2k^2 - 2b = 2(k^2 - b) = 1$. This is a contradiction (1 is not an even integer!). Therefore if $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$. □

PROBLEM 3. Let a and b be real numbers. Consider the statement: if a is less than every real number greater than b, then $a \leq b$.

1. State the contrapositive.

   Answer. If $a > b$, then a is larger than some real number c greater than b. □

2. Prove the statement (hint: how can you construct a number between a and b?)

   Proof. We prove the contrapositive. Suppose that $a > b$. Let $c = \frac{a+b}{2}$. Since c is the average of a and b it should be clear that $a > c > b$. To prove this formally, note that $c = \frac{a+b}{2} > \frac{b+b}{2} = b$ and $c = \frac{a+b}{2} < \frac{a+a}{2} = a$. Since $a > c > b$, this proves the contrapositive (and hence the original statement). □

PROBLEM 4. Consider the statement

A: There do not exist natural numbers m and n such that

$$\frac{4}{5} = \frac{1}{m} + \frac{1}{n}.$$

(a) Write down the negation of the statement A.

Answer. The negation is simply: There exist natural numbers m and n such that

$$\frac{4}{5} = \frac{1}{m} + \frac{1}{n}.$$

□

(b) Assume the statement in (a). We will consider various cases for m and n and show that each case is impossible.

(i) Consider the following case:

Case I: $m \geq 3$ and $n \geq 3$.

Show that this case is impossible (hint: how large can $\frac{1}{m} + \frac{1}{n}$ be?)

Proof. Case I: $m \geq 3$ and $n \geq 3$. Since $m \geq 3$ and $n \geq 3$, we have $\frac{1}{m} \leq \frac{1}{3}$ and $\frac{1}{n} \leq \frac{1}{3}$. Then $\frac{1}{m} + \frac{1}{n} \leq \frac{2}{3}$. Note that $\frac{2}{3} < \frac{4}{5}$. So it is impossible that $\frac{1}{m} + \frac{1}{n} = \frac{4}{5}$. □

(ii) What cases for m and n remain? Show that each of the remaining cases is also impossible.

Proof. If $m < 3$ or $n < 3$ then one of m or n is equal to 1 or 2. We consider these two cases.

Case II: One of m or n is equal to 1. In this case $\frac{1}{m} + \frac{1}{n} \geq 1 > \frac{4}{5}$. So it is impossible that $\frac{1}{m} + \frac{1}{n} = \frac{4}{5}$.

Case III: One of m or n is equal to 2. Without loss of generality, suppose that $m = 2$. Then

$$\frac{1}{n} = \frac{4}{5} - \frac{1}{2} = \frac{3}{10},$$

which is impossible. □

(c) Explain why your work in (b) proves the statement A. Which methods of proof were used here?

Answer. We assumed the negation of A and arrived at a contradiction (in every possible case for m and n). Therefore statement A must be true. This proof combined proof by contradiction with proof by cases. □

(d) (Extra) In contrast to what we've proved, a conjecture of Erdös and Straus states that for any integer $n \geq 2$ we can always write

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

for some $x, y, z \in \mathbb{N}$. To this day, the conjecture is unproven (but it is known for $n < 10^{14}$). Can you see how to write $\frac{4}{5}$ in the above way?

Answer.

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}.$$

□