Math 299 **Recitation 6** **Oct. 10, 2013**

PROBLEM 1. Let S be a set. Define a function $f : S \to \mathbb{R}$ to be a *topped out on* S if $\exists z \in \mathbb{R}$ such that $\forall s \in S$,

$$f(s) + 2 \leq z.$$

(a) Suppose $S = \{0, 1, 2\}$. Draw a picture of a function which is topped out on $\{0, 1, 2\}$.

Answer: Any function will do.

(b) Suppose $S = \{0, 1, 2\}$. Is it possible for a function $f : \{0, 1, 2\} \to \mathbb{R}$ to *not* be topped out on $\{0, 1, 2\}$?

Answer: Given any function f, take z to be the max value in the set

$$\{f(0) - 2, f(1) - 2, f(2) - 2\}$$

(c) Suppose $S = \mathbb{R}$. Draw a picture of a function which is topped out on $\mathbb{R}$.

Answer: The graph of any bounded function, e.g., a constant function.

(d) Suppose $S = \mathbb{R}$. Draw a picture of a function which is not topped out on $\mathbb{R}$.

Answer: The graph of any linear function with non-zero slope.

(e) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are each topped out on $\mathbb{R}$. Prove that

$$\begin{aligned} f + g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) + g(x) \end{aligned}$$

is topped out on $\mathbb{R}$.

Answer: Let z_f and z_g be the values determined by the 'topped out condition'. Then

$$(f(x) + g(x)) - 2 \leq z_f + z_g + 2.$$

PROBLEM 2. (Axioms for a 3 point geometry) Undefined terms: *point*, *line* and what it means for a point to be *on* a line. These satisfy the following axioms:

A1: *There are exactly 3 points.*
A2: *Each two distinct points are on exactly one line.*
A3: *Given two distinct lines, there exists a point which is on each of these two lines.*

Answer the following questions.

(a) Write down a model which satisfies these axioms. Clearly state what *point*, *line* and *on* mean in your model.

Answer: See attached.

(b) Write down a model which satisfies axioms A1 and A2, but not A3.

Answer: See attached.

(c) Write down a model which satisfies axioms A1 and A3, but not A2.

Answer: See attached.

(d) Suppose we included the following additional axiom

A4: *Not all of the points are on the same line.*

Give an example of a model which satisfies A1-A4.

Answer: See attached.

(e) Give an example of a model which satisfies A1-A3, but not A4.

Answer: See attached.

(f) (Optional) Prove by contradiction that any model satisfying axioms A1 and A2 must have at least some lines.

Answer: Suppose there are zero lines in a model satisfying A1 and A2. Let z_1, z_2 be points, which exist by A1. Then there are no line with z_1 and z_2 on it, this contradicts

A2.

(g) (Optional) Prove by contradiction that any model satisfying axioms A1-A4 cannot have exactly one line.

Answer: If there was one line, then by A4 there would be a point not on that line. This contradicts A2 with that point together with any other point.

(h) (Optional) Prove by contradiction that any model satisfying axioms A1-A4 cannot have exactly two lines.

Answer: Suppose a model had only two line ℓ_1 and ℓ_2. By A4, the line ℓ_1 does not contain all of the points, but by A3 it contains at least one point. Similarly, ℓ_2 does not contain all of the points, but it contains at least one point. This implies that ℓ_1 contains a point z_1 which is not on ℓ_2. Similarly, ℓ_2 contains a point z_2 which is not on ℓ_1. Since there are only two lines, there is no line between z_1 and z_2. This contradicts A2.

It can be shown that any model satisfying A1-A4 must have exactly 3 lines.