1. Find the number N such that $\forall n > N$ we have an inequality

$$\left|\frac{3n-1}{n+1} - 3\right| < \varepsilon$$

for given ε as follows:

(*Sol*) $\forall \varepsilon > 0$, $\exists N = \frac{4}{\varepsilon}$ such that $\forall n \geq N$,

$$\left|\frac{3n-1}{n+1} - 3\right| = \left|\frac{3n-1-3n-3}{n+1}\right| = \left|\frac{-4}{n+1}\right| = \frac{4}{n+1} < \frac{4}{n} < \frac{4}{N} = \varepsilon$$

(1) $\varepsilon = 0.1.$: $N = 40$, that is, for all $n \geq 40$, $\left|\frac{3n-1}{n+1} - 3\right| < \varepsilon$.

(2) $\varepsilon = 0.01.$: $N = 400$.

(3) $\varepsilon = 1 \times 10^{-5}.$: $N = 4 \times 10^5$.

2. By using the formal definition of the limit of the sequence prove the following:

(1) $\lim_{n\to\infty} \left(c + \frac{1}{n}\right) = c$ where c is a real number.

Proof. $\forall \varepsilon > 0$, $\exists N = 1/\varepsilon$ such that $\forall n > N$,

$$\left|\left(c + \frac{1}{n}\right) - c\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

(2) $\lim_{n\to\infty} \frac{1}{\sqrt{n+2}} = 0$.

Proof. $\forall \varepsilon > 0$, $\exists N = 1/\varepsilon^2$ such that $\forall n > N$,

$$\left|\frac{1}{\sqrt{n+2}} - 0\right| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} = \varepsilon$$

(3) $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$.

Proof. $\forall \varepsilon > 0$, $\exists N = \sqrt{2/\varepsilon}$ such that $\forall n > N$,

$$\left|\frac{n^2-1}{n^2+1} - 1\right| = \frac{2}{n^2+1} < \frac{2}{n^2} < \frac{2}{N^2} = \frac{2}{\left(\sqrt{\frac{2}{\varepsilon}}\right)^2} = \varepsilon$$

(4) $\lim_{n\to\infty} 3 + \frac{(-1)^{n+1}}{n} = 3$.

Proof. $\forall \varepsilon > 0$, $\exists N = 1/\varepsilon$ such that $\forall n > N$,

$$\left|3 + \frac{(-1)^{n+1}}{n} - 3\right| = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

3. **Proposition**: An upper bound b of a nonempty set $S \subseteq \mathbb{R}$ is the supremum of S if and only if $\forall \varepsilon > 0$, $\exists s \in S$ such that $b - \varepsilon < s$.

By using the proposition, prove the following statement.

Suppose that $S \subseteq \mathbb{R}$ is bounded above and that $b \in \mathbb{R}$ is an upper bound of S. Then $b = \sup S$ if and only if there exists a sequence (x_n) of elements in S converging to b.

Proof. ($\Rightarrow$) Assume that $b = \sup S$. By the proposition, for any $n \in \mathbb{N}$, $\exists x_n \in S$ such that

$$b - \frac{1}{n} < x_n.$$

Also, since b is an upper bound of S, $x_n \leq b$. Thus $b - \frac{1}{n} < x_n \leq b < b + \frac{1}{n}$, which implies that $\mid x_n - b \mid < \frac{1}{n}$. By the definition of convergence of limit, $\forall \varepsilon > 0$, $\exists N = 1/\varepsilon$ such that $\forall n > N$, $\mid x_n - b \mid < \frac{1}{n} < \frac{1}{N} = \varepsilon$. Therefore, the sequence (x_n) converges to b as required.

($\Leftarrow$) Suppose that there is a sequence (x_n) in S such that $\lim_{n\to\infty} x_n = b$. Then, by the definition of convergence of limit, for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\mid x_N - b \mid < \varepsilon$. So from this inequality, we have $-\varepsilon < x_N - b < \varepsilon$. So, in particular, we can get $b - \varepsilon < x_N$. Then, by the proposition, $b = \sup S$.

4. Suppose that $S \subseteq \mathbb{R}$ is nonempty and bounded above and let $-S = \{-x | x \in S\}$. Prove that $\inf(-S) = -\sup S$.

   Proof. To show that $-S$ is bounded below, let u be an upper bound of S, that is, $x \leq u$ for all $x \in S$. Then, $-x \geq -u$ for all $x \in S$. Thus, $-u$ is less than or equal to every element in a set $-S$. Therefore, there exists a lower bound $-u$ of $-S$. $-S$ is bounded below.

   Now, let's show that $\inf(-S) = -\sup(S)$. Let $a \in \mathbb{R}$ be a lower bound of $-S$. Then, $a \leq -x$ for all $x \in S$. Then, multiplying by -1 on both sides gives $x \leq -a$ for all $x \in S$. Thus $-a$ is an upper bound of S. Since $\sup S$ is a least upper bound, $\sup S \leq -a$. So $a \leq -supS$. Since a is any arbitrary lower bound of $-S$ and $-\sup S$ is greater than or equal to a, by the definition of infimum, $-\sup S = \inf(-S)$.

5. Let $s_n = n!/n^n$ for all $n \in \mathbb{N} \setminus \{0\}$. Prove that $\lim_{n\to\infty} s_n = 0$.

   Proof. The sequence s_n can be written as a product of fractions,

   $$s_n = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right).$$

   Note that each fraction is less than or equal to 1. So we have $\mid s_n - 0 \mid = s_n \leq \left(\frac{1}{n}\right) \times 1^{n-1} = \left(\frac{1}{n}\right)$. Therefore, $\forall \varepsilon > 0$, $\exists N = \varepsilon^{-1}$ such that $\forall n > N$, $\mid s_n - 0 \mid < 1/n < 1/N = \varepsilon$. That is, by the definition of convergence of limit, $\lim_{n\to\infty} s_n = 0$.

6. Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let $M = \{a \cdot b : a \in A \text{ and } b \in B\}$. Prove or disprove (provide a counterexample) that $\sup M = (\sup A) \cdot (\sup B)$.

   It is not true. A countexample would be $A = \{x \in \mathbb{R} | -2 \leq x < 1\}$ and $B = \{x \in \mathbb{R} | -3 \leq x < 1\}$.

7. Prove that if $\lim_{n\to\infty} a_n = 1$, then $\lim_{n\to\infty}(1 + a_n)^{-1} = 1/2$.

   Proof. Because $\lim_{n\to\infty} a_n = 1$, by the definition of limit, for $\varepsilon > 0$ there exists N_1 such that for all $n > N_1$ $\mid a_n - 1 \mid < \varepsilon$. Also, for $\varepsilon = 1$, there exists N_2 such that for all $n > N_2$, $\mid a_n - 1 \mid < \varepsilon$ which implies $a_n > 0$. Thus $\forall \varepsilon > 0$, $\exists N = \max\{N_1, N_2\}$ such that for all $n > N$,

   $$\left|\frac{1}{1 + a_n} - \frac{1}{2}\right| = \left|\frac{2 - 1 - a_n}{2(1 + a_n)}\right| = \frac{\mid 1 - a_n \mid}{2(1 + a_n)} < \mid 1 - a_n \mid < \varepsilon$$

   Therefore, $\lim_{n\to\infty}(1 + a_n)^{-1} = 1/2$.