Algebra Axioms. In Real Analysis, we work within the axiomatic system of real numbers: the set $\mathbb{R}$ along with the addition and multiplication operations $+, \cdot$, and the inequality relation $<$. We do not need to list or describe the elements of $\mathbb{R}$ directly; rather, anything we want to know about $\mathbb{R}$ will follow from Axioms 1–10.

We start with the axioms of the addition and multiplication operations, which include the commutative group axioms. For any $a, b, c \in \mathbb{R}$, we have:

1. Closure: $a + b \in \mathbb{R}$	1′. Closure: $ab \in \mathbb{R}$
2. Associativity: $(a + b) + c = a + (b + c)$	2′. Associativity: $(ab)c = a(bc)$
3. Identity element: $\exists 0,\ a + 0 = a$	3′. Identity element: $\exists 1 \neq 0,\ a1 = a$
4. Inverse: $\forall a, \exists\, b,\ a + b = 0$. Denote $b = -a$	4′. Inverse: $\forall a \neq 0, \exists\, b,\ ab = 1$ Denote $b = a^{-1}$
5. Commutativity: $a + b = b + a$	5′. Commutativity: $ab = ba$

6. Distributivity: $a(b + c) = ab + ac$

We define new operations in terms of the basic ones: subtraction $a - b$ means $a + (-b)$; and division a/b or $\frac{a}{b}$ means $a \cdot b^{-1}$.

We also have the axioms of inequality. For any $a, b, c, d \in \mathbb{R}$, we have:

7. Trichotomy: Exactly one of the following is true: $a < b,\ a = b,\ a > b$.
8. Compatibility of $<$ with $+$: If $a < b$ and $c < d$, then $a + c < b + d$.
9. Compatibility of $<$ with $\cdot$: If $a < b$ and $0 < c$, then $ac < bc$.

We define $a > b$ to mean $b < a$, and $a \leq b$ to mean $a < b$ or $a = b$.

Completeness. The final axiom gives a precise meaning to the idea that the real numbers have no holes or gaps, but rather form a continuum.

First, some definitions. We say a number $b \in \mathbb{R}$ is an *upper bound* for a set $S \subset \mathbb{R}$ whenever $x \leq b$ for all $x \in S$. Furthermore, ℓ is the *least upper bound* (or *supremum*) of S means $\ell \leq b$ for every upper bound b of S. We denote this as $\ell = \mathrm{lub}(S)$ or $\sup(S)$. Intuitively, the least upper bound is the "rightmost edge" of S on the real number line.

EXAMPLES: (i) Let $S = \mathbb{N} \subset \mathbb{R}$. Then S is an unbounded set having *no* upper bounds, and hence no least upper bound.

(ii) Let $S = \{0.9, 0.99, 0.999, \ldots\}$. Then every number $b \geq 1$ is an upper bound of S, and $\ell = \sup(S) = 1$ is the least upper bound. It makes no difference whether 1 is in the set or not: $S \cup \{1\}$ has the same upper bounds as S, and $\sup(S \cup \{1\}) = 1$.

(iii) Let $S = \{x \in \mathbb{R} \mid x^2 < 2\}$. Some upper bounds for S are upper approximations to $\sqrt{2}$, like $a = 1.5, 1.42, 1.415, \ldots$. The least upper bound is $\sup(S) = \sqrt{2}$ itself, which is a way of producing this irrational number without assuming it exists.

10. Axiom of Completeness: Any set $S \subset \mathbb{R}$ which has an upper bound, also has a least upper bound in the reals: $\sup(S) \in \mathbb{R}$.

Note that this axiom fails for the rational numbers $\mathbb{Q}$, and this is their main difference from the real numbers. For example, the set in Example (iii) above has upper bounds in $\mathbb{R}$ and in $\mathbb{Q}$, but it has a least upper bound *only* in $\mathbb{R}$: in the rationals $\mathbb{Q}$, there is a "hole" where $\sup(S) = \sqrt{2}$ would be, since $\sqrt{2}$ is *not* rational.

Algebra Propositions. All the usual facts of algebra (including inequalities) can be deduced from Axioms 1–9. Throughout, we implicitly use Axioms $2, 2'$ to write expressions like $a+b+c$ instead of $(a+b)+c$, and abc instead of $(ab)c$.

PROPOSITION 1 (Mulitplication by zero): $0a = 0$.
Proof. By Axioms 3 and 6, we have: $0a = (0+0)a = 0a + 0a$. Adding $-0a$ to the left and right sides of this equality, we get: $0a - 0a = 0a + 0a - 0a$, which we can simplify by Axioms 4 and 3 to $0 = 0a$ as desired.

PROPOSITION 2 (Multiplication with signs):

$$\text{(i) } -(-a) = a, \quad \text{(ii) } (-a)b = -(ab), \quad \text{(iii) } (-a)(-b) = ab.$$

Proof. (i) By Axiom 4, $0 = (-a) - (-a)$. Adding a to both sides gives: $a = a + (-a) - (-a) = 0 - (-a) = -(-a)$.

(ii) We have: $(-a)b + ab = (-a+a)b = 0b = 0$ by Prop. 1. Switching the sides: $0 = (-a)b+ab$, and adding $-(ab)$ to both sides: $-(ab) = (-a)b+ab-(ab) = (-a)b+0 = (-a)b$.

(iii) Applying (ii) twice, we have: $(-a)(-b) = -(a(-b)) = -((-b)a) = -(-(ba)) = -(-(ab))$. Thus $(-a)(-b) = -(-(ab)) = ab$ by (i).

PROPOSITION 3. If $a < b$, then $-b < -a$.
Proof. Let $a < b$. Using Axiom 8, we add $-a-b$ to both sides, getting: $a + (-a-b) < b + (-a-b)$. Simplifying the left and right sides by Axioms 2, 3, 4, 5 gives $-b < -a$.

PROPOSITION 4. $0 < 1$.
Proof. Surprisingly, this is not immediate. Suppose for a contradiction that $0 \not< 1$. By Axiom 7, this means $0 \geq 1$, but Axiom $3'$ says $0 \neq 1$. Thus $0 > 1$, and by Prop. 3, $0 < -1$, so by Axiom 9, $0(-1) < (-1)(-1)$. By Prop. 1, $0(-1) = 0$, and by Prop. 2, $(-1)(-1) = (1)(1) = 1$, which means $0 < 1$. But we already saw $0 > 1$, so this contradicts the uniqueness part of Axiom 7. This contradiction shows $0 < 1$.

PROPOSITION 5 (Transitivity of $<$): If $a < b$ and $b < c$, then $a < c$.
Proof. Suppose $a < b < c$. By Axiom 8, we can subtract a from the first inequality to get $0 < b - a$, and subtract b from the second inequality to get $0 < c - b$. Adding these two inequalities: $0 < (b-a) + (c-b) = c - a$. Adding a to ths inequality gives $a < (c-a) + a = c$.

DEFINITION: $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

PROPOSITION 6: For $a, b \in \mathbb{R}$: (i) $|ab| = |a|\,|b|$. (ii) $|a+b| \leq |a| + |b|$;
(iii) (Triangle inequality) For $x, y, z \in \mathbb{R}$, $|x - z| \leq |x-y| + |y-z|$.

Proof. (i) If $a, b \geq 0$, then $ab > 0$ by Axiom 9, and by definition $|ab| = ab = |a|\,|b|$. If $b < 0 \leq a$, then $a, -b \geq 0$ and $-(ab) = a(-b) > 0$, so $ab < 0$; thus $|ab| = -ab = a(-b) = |a|\,|b|$ by Prop. 2. Similarly for $a < 0 \leq b$. If $a, b < 0$, then $ab = (-a)(-b) > 0$ and $|ab| = (-a)(-b) = |a|\,|b|$. Axiom 7 guarantees that we have considered all possible cases.

(ii) We have $|x| = \max\{x, -x\}$, so $|a+b| = \max\{a+b, -a-b\}$, whereas we easily see: $|a| + |b| = \max\{a+b, a-b, -a+b, -a-b\}$. The larger set clearly has a larger maximum, so $|a+b| \leq |a| + |b|$.

(iii) This follows from (ii) taking $a = x - y$, $b = y - z$, so that $a + b = x - z$.

Limits. Consider an infinite sequence $(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots)$ with $a_i \in \mathbb{R}$.

DEFINITION: We say (a_n) converges to the number L, written $\lim_{n\to\infty} a_n = L$, meaning that for any error bound $\epsilon > 0$, there exists a threshold $N \in \mathbb{N}$ (depending on ϵ) such that $n \geq N$ forces a_n into the error interval $L - \epsilon < a_n < L + \epsilon$. In symbols:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ n > N \Rightarrow |a_n - L| \leq \epsilon.$$

PROP 7: If $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = M$, then (i) $\lim\limits_{n\to\infty} a_n + b_n = L + M$; (ii) $\lim\limits_{n\to\infty} a_n b_n = LM$.
Proof. By the definition of the limits in the hypothesis, for any $\epsilon_1 > 0$, there is N_1 such that $n \geq N_1 \Rightarrow |a_n - L| < \epsilon_1$; and for any $\epsilon_2 > 0$, there is N_2 such that $n \geq N_2 \Rightarrow |b_n - M| < \epsilon_2$.

(i) Now let $\epsilon > 0$ and take $\epsilon_1 = \epsilon_2 = \frac{1}{2}\epsilon$. Then taking $N = \max(N_1, N_2)$ above, for any $n \geq N$ we have:

$$\begin{array}{rcll} |(a_n + b_n) - (L + M)| & = & |(a_n - L) + (b_n - M)| & \\ & \leq & |a_n - L| + |b_n - M| & \text{by Prop 6, Triangle Inequality} \\ & < & \epsilon_1 + \epsilon_2 = \epsilon & \text{since } n \geq N \geq N_1, N_2. \end{array}$$

(ii) Now let $\epsilon > 0$, and take $\epsilon_1 = \frac{\epsilon}{2|L|+1}$ and $\epsilon_2 = \frac{\epsilon}{2|M|+1}$. Also take $\epsilon_1' = \frac{1}{2}$, so that $n \geq N_1' \Rightarrow |a_n - L| < \frac{1}{2}$, and $|a_n| < |L| + \frac{1}{2}$. Then taking $N = \max(N_1, N_2, N_1')$, for any $n \geq N$ we have:

$$\begin{array}{rcll} |a_n b_n - LM| & = & |a_n b_n - a_n M + a_n M - LM| & \\ & = & |a_n(b_n - M) + (a_n - L)M| & \\ & \leq & |a_n|\,|b_n - M| + |a_n - L|\,|M| & \text{by Prop 6(ii)} \\ & < & |a_n|\epsilon_2 + \epsilon_1 |M| & \text{since } n \geq N \geq N_1, N_2 \\ & < & (|L| + \frac{1}{2})\dfrac{\epsilon}{2|L|+1} + \dfrac{\epsilon}{2|M|+1}|M| & \text{since } n \geq N_1' \\ & < & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon & \text{since } \frac{x+\frac{1}{2}}{2x+1} = \frac{1}{2},\ \frac{x}{2x+1} < \frac{1}{2} \text{ if } x \geq 0. \end{array}$$

Infinite Approximations. We use limits to handle real numbers which we cannot define directly, but only through approximations, such as a derivative, an infinite series, or an infinite decimal.

PROPOSITION 8: The derivative of the function $f(x) = \frac{1}{x}$ at $x = 2$, namely the tangent slope of $y = \frac{1}{x}$ at $(x, y) = (2, \frac{1}{2})$, is: $f'(2) = \frac{dy}{dx}|_{x=2} = -\frac{1}{4}$.
Rough Draft of Proof. The tangent line is very difficult to construct, because it is defined as touching the curve at only the one point $(2, \frac{1}{2})$, and one point does not define a line. However, it is easy to construct secant lines, which cut the curve at two nearby points, $(2, \frac{1}{2})$ and $(2+\frac{1}{n}, \frac{1}{2+\frac{1}{n}})$. The great idea of differential calculus is that the tangent slope is the limit of secant slopes:

$$f'(2) = \lim_{n\to\infty} \frac{f(2+\frac{1}{n}) - f(2)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{2+\frac{1}{n}} - \frac{1}{2}}{\frac{1}{n}}.$$

We want to show this limit converges to $-\frac{1}{4}$, and we work backwards from the desired conclusion, that the distance between the sequence and the limit is within any desired error tolerance $\epsilon > 0$:

$$\left| \frac{\frac{1}{2+\frac{1}{n}} - \frac{1}{2}}{\frac{1}{n}} - \left(-\frac{1}{4}\right) \right| < \epsilon$$

Simplifying the lefthand side, this becomes: $\left| -\frac{n}{4n+2} + \frac{1}{4} \right| = \frac{1}{8n+4} < \epsilon$. Now, $\frac{1}{8n+4} < \frac{1}{8n}$, so it is enough to show $\frac{1}{8n} < \epsilon$. Solving this last inequality for n gives: $n \geq \frac{1}{8\epsilon}$. Thus, to guarantee the desired conclusion, we only need that $n \geq N$, where N is any integer greater than $\frac{1}{8\epsilon}$. (If ϵ is a very small error tolerance, this N is very large, but we know there is an integer larger than any given real number.)

Final Proof. We want to compute the derivative of $f(x)$ at $x = 2$, and show:

$$f'(2) = \lim_{n\to\infty} \frac{f(2+\frac{1}{n}) - f(2)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{2+\frac{1}{n}} - \frac{1}{2}}{\frac{1}{n}} = -\frac{1}{4}\,.$$

Given $\epsilon > 0$, take an integer $N > \frac{1}{8\epsilon}$. Then for $n \geq N$, we compute:

$$\left| \frac{\frac{1}{2+\frac{1}{n}} - \frac{1}{2}}{\frac{1}{n}} - \left(-\frac{1}{4}\right) \right| = \left| -\frac{n}{4n+2} + \frac{1}{4} \right| = \frac{1}{8n+4} < \frac{1}{8n} \leq \frac{1}{8N} < \frac{1}{8\left(\frac{1}{8\epsilon}\right)} = \epsilon.$$

By definition, this proves the convergence of the limit.

THEOREM 9: Suppose the sequence (a_n) is increasing, with an upper bound b: that is, $a_1 \leq a_2 \leq a_3 \leq \cdots \leq b$. Then (a_n) converges to $L = \sup\{a_n \mid n \geq 1\}$.

Proof. The least upper bound $L = \sup\{a_n\}$ exists by Axiom 10. Then L is an upper bound, so $a_n \leq L < L + \epsilon$ for all n and all $\epsilon > 0$.

Since L is the *least* upper bound, we know that $L - \epsilon$ is not an upper bound of (a_n) for any $\epsilon > 0$. This can only be if $L - \epsilon < a_N$ for some N, and since $a_N \leq a_n$ for all $n \geq N$, we have:

$$L - \epsilon < a_n < L + \epsilon,$$

namely $|L - a_n| < \epsilon$. In summary, for any $\epsilon > 0$, there is some N such that $n \geq N$ implies $|L - a_n| < \epsilon$. This is precisely the definition of $\lim_{n\to\infty} a_n = L$.

PROPOSITION 10 (Sum of geometric series): For a fixed $x \in \mathbb{R}$, define the geometric series (s_n) by: $s_n = 1 + x + x^2 + \cdots + x^n$. If $|x| < 1$, then (s_n) converges to $L = \frac{1}{1-x}$.

Proof. Recall that we proved (by induction in HW due 10/23) the formula:

$$s_n = \frac{1 - x^{n+1}}{1 - x}\,.$$

In the case that $0 \leq x < 1$, we clearly have $s_1 \leq s_2 \leq \cdots \leq \frac{1}{1-x}$, so that (s_n) is an increasing bounded sequence, and it converges by the Theorem.

We leave the exact value of the limit, and the case $-1 < x < 0$, as an exercise.

THEOREM (Decimal expansions): Let $(d_n) = (d_1, d_2, \ldots)$ be a sequence of digits, with $d_n \in \{0, 1, \ldots, 9\}$. Define an increasing sequence (a_n) by:

$$a_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_2}{10^3} + \cdots + \frac{d_n}{10^n}\,,$$

namely the n-digit decimal $0.d_1d_2\dots d_n$. Then (a_n) converges to a unique real number, the infinite decimal $0.d_1d_2d_3\dots$.

Proof. We clearly have $\frac{d_n}{10^n} < \frac{10}{10^n}$, so that $a_n < b_n$ for the geometric series:

$$b_n = 1 + \tfrac{1}{10} + \left(\tfrac{1}{10}\right)^2 + \cdots + \left(\tfrac{1}{10}\right)^n .$$

Now, by the previous proposition, the geometric series (b_n) has the upper bound $b = \frac{1}{1-x}$ for $x = \frac{1}{10}$, and we have $a_n < b_n < b$, so that (a_n) is an increasing bounded sequence, and converges to some real number.

Proposition 11: There is positive real number ℓ with $\ell^2 = 2$; that is, $\sqrt{2} \in \mathbb{R}$.

Proof. Let $S = \{x \in \mathbb{R} \mid x^2 < 2\}$. Now, if $x \in S$ and $1 < x$, then $x < x^2 < 2$, so clearly $b = 2$ is an upper bound of S. By the Completeness Axiom, S has a least upper bound $\ell = \sup(S) \leq 2$.

We will show that $\ell^2 = 2$ by contradiction. First, suppose $\ell^2 < 2$. Then we may choose ϵ with $0 < \epsilon < \frac{1}{5}(2 - \ell^2)$, and also $\epsilon < 1$. We compute:

$$\begin{array}{rcl} (\ell+\epsilon)^2 & = & \ell^2 + (2\ell{+}\epsilon)\epsilon \\ & < & 2 + (2(2){+}1)\epsilon \\ & < & 2 + (5)(\frac{1}{5})(2{-}\ell^2) = 2. \end{array}$$

By definition, this means $\ell + \epsilon \in S$ with $\epsilon > 0$; but ℓ is an upper bound of S, so $\ell + \epsilon \leq \ell$ with $\epsilon > 0$. This contradiction shows $\ell^2 < 2$ is impossible.

Next, suppose $\ell^2 > 2$. Then we may choose ϵ with $0 < \epsilon < \frac{1}{4}(\ell^2 - 2)$, and also $\epsilon < 1$. We compute:

$$\begin{array}{rcl} (\ell-\epsilon)^2 & = & \ell^2 - 2\ell\epsilon + \epsilon^2 \\ & > & 2 - 2(2)\epsilon + 0 \\ & > & 2 - (4)(\frac{1}{4})(\ell^2 - 2) = 2. \end{array}$$

Thus, for any $x \in S$, we have $x^2 < 2 < (\ell - \epsilon)^2$, with $\ell - \epsilon > \ell - 1 \geq 0$. Now we apply the exercise that if $x^2 < y^2$ with $y > 0$, then $x < y$: letting $y = \ell - \epsilon$, this means $x < \ell - \epsilon$. Thus $\ell - \epsilon$ is an upper bound of S, but ℓ is the least upper bound, so $\ell \leq \ell - \epsilon$ with $\epsilon > 0$. This contradiction shows $\ell^2 > 2$ is impossible.

Problems

Prove the following statements using the above Axioms and Propositions (saying which results you use).

1. If $0 < a < b$, then $b^{-1} < a^{-1}$.

2. For $y > 0$, if $x^2 < y^2$, then $x < y$.

3a. Using the formal definiton, prove: $\lim\limits_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2-1}{\frac{1}{n}} = 2$.
Hint: For the rough draft, work backward from the conclusion $|a_n - L| < \epsilon$.
b. What does the above limit mean in calculus? *Hint:* It concerns the behavior of the function $f(x) = x^2$ near $x = 1$.

c. Prove that $\lim\limits_{n\to\infty} \frac{2n^2-1}{3n^2+n} = 2/3$. *Hint:* Avoid false inequalities like $|1-2n| \overset{?}{<} |1|$ or $\frac{1}{2n^2-1} \overset{?}{<} \frac{1}{2n^2}$. You may use that $an - c > \frac{a}{2}n$ for $a > 0$ and large n (how large?).

4a. There is no largest element of $\mathbb{R}$. Hint: Contradiction.
b. There is no smallest element of the positive reals $\mathbb{R}_{>0}$.
c. If $|x| < \epsilon$ for all $\epsilon > 0$, then $x = 0$. *Hint:* Contradiction.

5. The limit $\lim_{n\to\infty} a_n$ converges to at most one value. That is, if (a_n) converges to L and also to L', then $L = L'$.

Hint: This is not just a matter of writing $L = \lim_{n\to\infty} a_n = L'$, since the whole point is to prove the limit in the middle is a well-defined, unambiguous value. Rather, write out the definition of $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = L'$ and use the Triangle Inequality above to prove that $|L - L'| < \epsilon$ for every $\epsilon > 0$. Why does this give the conclusion?

6. If (a_n) is a convergent sequence, then the sequence is bounded. That is, if $\lim_{n\to\infty} a_n = L$, then there is some B with $a_n \le B$ for all n.

7. Consider the sequence $a_n = n^2$.
a. Prove that $\lim_{n\to\infty} a_n = \infty$. This means that for every bound $B \in \mathbb{R}$, there is a threshold N such that $n \ge$N implies that $a_n > B$. That is, a_n goes above any bound for large enough values of N.
b. Prove that (a_n) is divergent; that is, (a_n) does not converge to any value $L \in \mathbb{R}$, meaning $\lim_{n\to\infty} a_n = L$ is *false.*

Hint: This does not follow immediately from part (a): you must show the definition of $\lim_{n\to\infty} a_n = \infty$ implies the negation of the definition of $\lim_{n\to\infty} a_n = L$, for any L. This means that there is an error bound, say $\epsilon = 1$, such that no matter how large $n \ge N$ is, the sequence a_n is not forced within the error bound $|a_n - L| < \epsilon$.

8. For each statement (a), (b) below, either prove it is true, or find a counterexample and prove its properties. Let (a_n) and (b_n) be any two real sequences, and define their sum (c_n) by $c_n = a_n + b_n$.

a. If (a_n) and (b_n) are convergent, then (c_n) is convergent. Hint: We discussed this situation in class; see also Beck Prop. 10.23.

b. If (c_n) is convergent, then (a_n) and (b_n) are convergent. Hint: Is there any way (a_n) and (b_n) could fluctuate around, even though (c_n) is converging to some L?

9. Use the geometric series formula $\lim_{n\to\infty}(1+x+x^2+\cdots+x^n) = \frac{1}{1-x}$ to find a fractional form $\frac{a}{b}$ for the repeating decimal $0.7343434\cdots = 0.7\overline{34}$. *Hint:* Write the decimal as 0.7 plus a multiple of a geometric series with $x = \frac{1}{100}$.

10. For bounded subsets $S, T \subset \mathbb{R}$, and $U = \{x + y \mid x \in S, y \in T\}$, we have $\sup(U) = \sup(S) + \sup(T)$.

11. Define a *lower bound* for a set $S \subset \mathbb{R}$ to mean some $b \in \mathbb{R}$ with $b \leq x$ for all $x \in S$. The *greatest lower bound* or *infimum* of S, denoted $\mathrm{glb}(S)$ or $\inf(S)$, means a lower bound g with $g \geq b$ for all lower bounds b. Denote $-S = \{-x \mid x \in S\}$.

a. The number b is a lower bound of S if and only if $-b$ is an upper bound of $-S$.

b. $\inf(S) = -\sup(-S)$.

c. Any set S having a lower bound b has a greatest lower bound $g = \inf(S) \in \mathbb{R}$.